UNIVERSIDAD DISTRITAL FRANCISCO JOSE DE CALDAS

Low Dimensional systems and anyonic statistics

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Declaration of Authorship

I, Yefferson Alexander Sierra Pineda , declare that this thesis titled, 'Low Dimensional systems and anyonic statistics' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a Bachelor degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Abstract

Faculty of Science and Education
Department of physics

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Anyonic statistics is a new knowledge in physics. It is based on the property of a configuration space to be multiply connected. This always happens in low dimensional systems, that is systems in two space dimensions. Also the anyonic statistics is based on the indistinguishability of particles. As a result of a symmetry exchange, that is, to exchange the position of two particles. In two dimensions the wavefunction acquires an non-trivial phase which is not the same as in the bosonic or fermionic statistics. This phase can be interpreted as an introduction of a gauge transformation in the lagrangian describing this system. Therefore, the anyons are particles that possess an extra type of "interaction". Anyons are conceptualized as a charged particle with a flux associated. Thus quantum mechanics change for anyonic particles. The only possible phenomenological application of the anyonic statistics is the description of the **FQHE** where anyons are conceived as quasi-holes or quasi-particles, with fractional charge. Anyons then have reality in the sense that anyon particles are observable by the **FQHE** experiments.

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Dedicated to my parents

Chapter 1

Introduction

1.1 A Pedagogic Introduction to Anyons

In order to give a soft introduction to the main subject, the best option is to use an approach based on analogies, since good arguments (true statements) can be given by this way (Juthe, 2005). This introduction to the theory is shown, in order to clear up the main scheme. For obvious reasons the physical properties are used in a qualitative way.

1.1.1 Statistics

In nature, particles can be classified in two families: bosons and fermions. The physical property which allows such classification is the intrinsic angular momentum. This angular momentum, named spin, is very different from the classical one. The first family of particles are characterized by having integer spin. That is, an intrinsic angular momentum given in units of $\hbar$. This family of particles are called bosons. On the other hand, fermions posses an intrinsic angular momentum given in half-integer units of $\hbar$. It happens that spin has a quantum mechanical nature, then quantum mechanics must be used in order to understand this. It is important to mention here that usually one deals with the z-component of the spin.

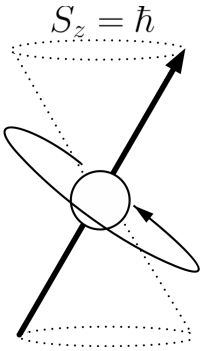


FIGURE 1.1: Example of spin for Bosons

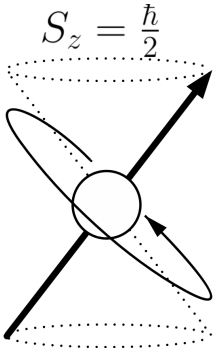


FIGURE 1.2: Example of spin for Fermions

As can be seen in Figure 1.1 and Figure 1.2, the spin just tell us about the possible projection of angular momentum spin on the z-component.

When dealing with big number of particles individual descriptions of each particle become cumbersome and unpractical. In this case the physical phenomena is described in terms of the collective response of the system, by means of an statistical description. Imagine for example the traffic. It would be almost impossible to describe the dynamics of each particular car.

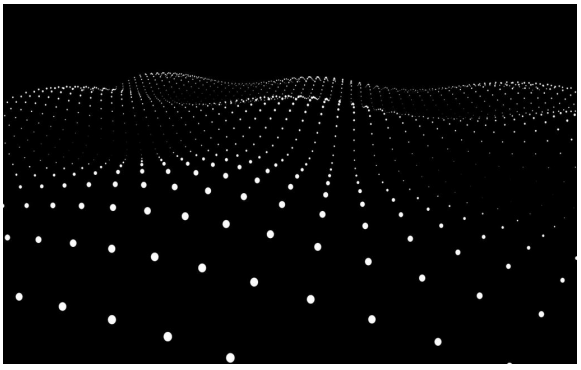


FIGURE 1.3: A big number of Particles (js.postbit.com, 2016)

In describing a system of this kind, the interaction between particles plays an important role and then particles behave in a random way. As a result of this, one has to study that big number of particles as a whole. In describing this number of particles, one uses a function named distribution function. In general the distribution function gives account for probabilities of particles to behave in some way, or another. A good way to imagine this is to look at a street; a popular street. As time passes on a normal day, is very probable that if the street is very popular people transits trough it.



FIGURE 1.4: Shibuya Crossing(for91days.com, 2016)

A distribution function then does not explain each passerby behavior, instead explain the whole people behavior in that street, by assigning a given probability to the bulk of people.

This can be seen on the famous Shibuya crossing in Japan Figure 1.4 where people walk toward each other without hitting. Then the most probable state of that street is to have people walking and no accidents for example. Here is where the use of analogies can become handy. As people in the aforementioned example can be of two types namely male persons or female persons, the whole behavior of the people would change if they are females or males for example. In the same way the complete behavior of a system of particles is determined by the spin of the particles.

The full behavior of the particles changes according to something named statistics. When particles are bosons the behavior of the whole system is described by the so called Bose-Einstein statistics. When particles are fermions the behavior of the whole system is described by the Fermi-Dirac statistics. One may wonder about what is the difference in the behavior of the system, this difference is clear when the properties of these systems are shown.

1. **Bosons:** the main property of systems of this kind is that, particles tend to condensate on the lowest state, this is using the example with people, when all of the people in a street behave in the same way.

2. **Fermions:** in this systems, particles feel some kind of repulsion which is named the Pauli exclusion principle. This principle states that two particles cannot be on the same state. In the example with people imagine people who don not like each other and then behave in different way.

1.1.2 Indistinguishability and exchange

On treating systems of a very high number of particles, one does not know what kind of particles they are (the family to which they belong). And furthermore, all particles are identical so that one can not distinguish between two particles. A way to see to which family the particles belong is to make an exchange. In the case of a high number of people, suppose that all of them are dressed in the same way, therefore they seem to be identical if the number of people is big enough. Then, if we choose two persons and make them to exchange their positions one can obtain two results: that the overall behavior of the people does not change at all, or that they behave in very different ways.

The change of this behavior is the same for a system of identical particles. If under an exchange of two particles the state changes, then they obeys different statistics(Bose-Einstein or Fermi-Dirac). This change in the behavior of the full ensemble is acknowledged as the spin statistics theorem, and relates the spin of a particle with the statistics that they obey.

1.1.3 Other possibilities for behavior

On treating this systems as mentioned before, we saw that particles behave on different ways. That is, they obey different statistics. Only two types of statistics where shown, because these are the only possible types of behavior. This is a true affirmation but only in our three dimensional world. In two dimensions this changes, and offers a new type of behavior. To see this, let us consider a two dimensional universe where there is a two dimensional earth, and two dimensional living beings. This world will be named the flatland following the popular book from Abbott (2006). In our two dimensional world let us suppose that the living beings discovered physics. Thus, in this two dimensional world physical systems are entirely two dimensional. Two dimensional physical systems will be named *Low dimensional systems*.

As described before particles belong to two different families. This are bosons or fermions. An we see how this particles can be identified on a system of a high number of particles by just exchanging two particles and looking for the result. Suppose that our two dimensional scientists in the flatland study a low dimensional system with a high number of particles. What

would be their result for describing the behavior of this particles? Would they have the same differentiation for particles? That is, to have two kind of families ?

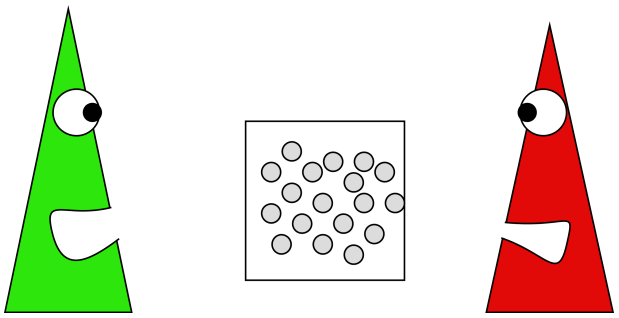


FIGURE 1.5: Flatland scientists studying a two dimensional ensemble of particles

To answer this question in a qualitative manner. Let us use the example with people as made before. Suppose that we have a very high number of persons in a park for example, but in this case this persons belong to the flatland, therefore they are two dimensional persons. As before if all this persons are dressed in the same way, and there is a big number of persons they would seem identical and indistinguishable. In the aforementioned example this persons exchange their positions and because of this they as a whole change their behavior. In this case, if we prompt two persons to exchange their position, we would have a big problem. In this world they find a point where they can not pass each other (in order to make the exchange), that is, the point where they coincide. As a result of this the whole problem of finding a family to this two dimensional beings becomes cumbersome. The reader might counter this argument explaining that in three dimensions there is also a point in space where two persons can coincide, but this can be solved by just passing over or under the other person, while in two dimensions that is just impossible. There is always a point where they cannot pass trough each other.

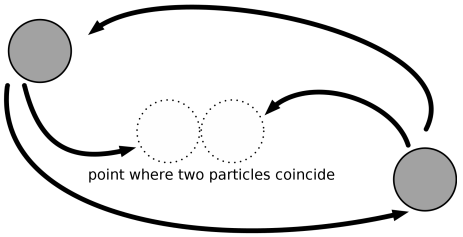


FIGURE 1.6: Low dimensional particles exchanging their position

This coincidence point makes the whole persons behavior to change, sometimes they behave as if nothing has happened and behave in the same way. And on other times they just behave in very different ways, thus this new type of behavior is just in between the two types of possible

known behaviors. To this new behavior scientist have named it anyonic statistics (since they can behave in any possible way in between two well known statistics). Returning to the study of particles, the particles obeying anyonic statistics are named anyons and they are the main subject of this thesis.

1.1.4 What about the flatland?

It is a matter of fact that this new type of behavior only happens in low dimensional systems. Furthermore, this new behavior happens where we have particles that can not pass trough each other. This kind of spaces are named multiply-connected spaces, and they have the property that there is a singular point. A way to understand this is to imagine a table which has hole.

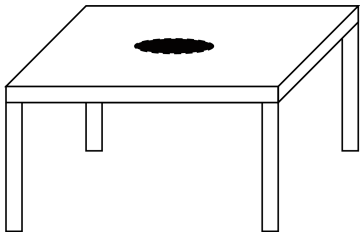


FIGURE 1.7: Table with a hole

The top of the table could be thought as a two dimensional space. To idealize two dimensional objects, imagine that we use a marker pen on it. We draw a line on that table, and then draw a second line on the table. Now if we try to draw lines with the same points in between both lines there would be a discontinuity in the hole, since some lines we try to draw there are discontinuous in that point.

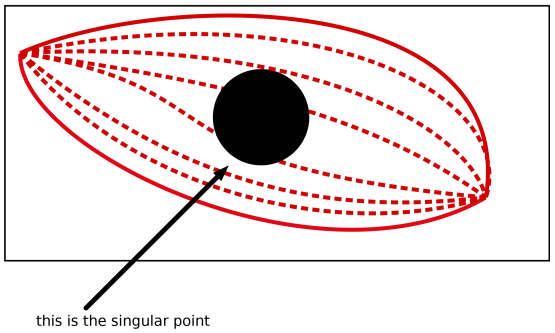


FIGURE 1.8: Table with a hole and some marks of a pen on it.

This is an easy way to idealize multiply-connected spaces. In fact only multiply connected spaces produce anyons.

1.1.5 Quantum Mechanics of lines

As a crucial way to understand the type of family to whom the particles belong, we must use spin. As spin has quantum mechanical nature, quantum mechanics must be used here. There exists some ways of quantum mechanics, since we are dealing with lines on two dimensional multiply-connected spaces. A way to quantize lines in quantum mechanics is to use the path integral formalism. It replaces the classical notion of a single, unique trajectory (line) for a system, with a sum over an infinity of possible trajectories. This means, that description of the behavior of systems is based on the trajectories. But, what about our space which has some holes? must we exclude the paths which transverse the holes? It happens that for simplicity and to solve the problem (a good description of quantum phenomena), there must be some trajectories that can be painted continuously on the table. This trajectories can be grouped on some types of trajectories. That is, there must exist for example trajectories which do not have the hole in between, and there must exists some trajectories which have a hole in between, also there must be a trajectory that wind around the hole, and so on. The fact is that quantum mechanics of lines(the path integral formalism) must be used, taking into account these different groups of lines.

1.1.6 Summary and final remarks

Up to now, the description of particles into families was made in order to give an analogy from the usual classification for particles. And from this the main problem of this thesis was told, based on the behavior of a system of identical particles. And from the exchange between particles. Into that, the low dimensional property was treated by making an useful analogy with the flatland and the physics in two dimensions. It is important to say that this analogy gives us answers by making use of a topological property of topological spaces to be multiply-connected. And to group the different kind of paths that can be deformed into each other to use the path integral quantization. To end this introduction there must be said that the full description of anyons depends upon some fundamental properties (almost like physical principles) which where treated in a soft way. They are:

1. Anyons must be low dimensional systems.
2. Anyons arise only in multiply-connected spaces.
3. A crucial way to understand anyons is by the use of paths.
4. Anyons arise in systems of indistinguishable particles.

The full theory of anyons will be developed based on the aforementioned properties, so its understanding is really important. The next section is an historical approach to the main subject, in order to clear up the arise and development of this theory.

1.2 History of fractional statistics

In the early days of quantum physics, the mathematical formalism of the theory was still a complete mistery. It was known that Heisenberg and Schrodinger used different representations of similar kinds of systems to describe them. These representations where denominated Heisenberg representation and Schrodinger representation, respectively. The first one is associated to the canonical commutation relationships and the phase space, meanwhile the later one is related to the representation of x,p as differential operators and the configuration space. Although both of them described the same physics, they were completely different at that time, and it was still unknown how to link them. Later on, with the formulation of quantum mechanics at a rigorous mathematical level proposed by John Von Neuman and Marshall Stone it was understood that they were related. This was stated as the Stone-Von Neuman theorem(Neumann, 1931, 1932; Stone, 1930, 1932). After the rigorous formulation of quantum mechanics of simple systems(Neumann, 1955) Heisenberg and Dirac proposed a quantum theory of indistinguishable systems of particles(Dirac, 1926, 1981; Heisenberg, 1985). They noted that the operators representing observables in suchs systems must be symmetric under any interchange of particle labels, since non-symmetric observables would allow an observer to distinguish between particles. This statement was the key to the correct quantum theory, because symmetric operators preserve the symmetry properties of the wave functions. This is important since the symmetry of the wave function can be associated with certain properties of a system which will be discussed later. All this development of the quantum theory was taking place in the 1920-1940 period, but not only this branch of physics was growing, also statistical mechanics was taking a very important place. About this time two important quantum descriptions of systems of particles where discovered: the Fermi-Dirac statistics and the Bose-Einstein statistics. The first one at that time was discovered by treating a system of electrons(Dirac, 1926; Fermi, 1926), while the second one was discovered by treating a system of photons (Bose, 1924; Einstein, 1924). Also the discovery of the spin in 1923 helped to understand the quantum theory of many particle physics since as it is well known all particles possesses integer-spin or half-integer spin (not true in two dimensional spaces). Indeed the spin of a system of particles was related to the statistics that the system obeys, and were Pauli and Fierz the ones who showed that relationship (Fierz, 1939; Pauli, 1940). Later this was called the Spin-Statistics theorem.

Theorem 1.1 (**Spin-Statistics Theorem**). *Particles with integer spin follow Bose– Einstein statistics (symmetric wave function) and are called bosons, while particles with half-integer spin follow Fermi–Dirac statistics (antisymmetric wave function) an are called fermions (Auletta, Fortunato, & Parisi, 2009, p. 249).*

Two important features arise at this point. First, as we can see Theorem 1.1 implies the relation between the statistics that a system obeys and the spin of its particles. Second, Theorem 1.1 implies something that arises when two particles in a system "exchange" their positions which is named the exchange interaction. The exchange interaction is a pure quantum mechanical effect that arise in a system of identical particles. This is an effective interaction that acts on the states of the particles. If the particles under exchange are bosons (particles with integer spin) they tend to condensate on the ground state which is denominated *Bose condensates*. On the other hand, if the particles under exchange are instead fermions (particles with half-integer spin) they can not be in the same single state which is denominated *Pauli Exclusion Principle* (Le Bellac, 2011, p. 441).

In treating a quantum system of identical particles, as has been said before, one can exchange particles. Apart from the resulting interaction, this induces a transformation on the wave function of n identical particles. Let the function representing the system $\Psi : \mathcal{Q} \to \mathbb{C}$ be

$$\Psi = \Psi(q_1, q_2, ..., q_i, ..., q_j, ..., q_n). \tag{1.1}$$

The transformation mentioned is called "Symmetry exchange" and it deals with the "exchange" of two particles in a system of identical particles. One can not imagine how two particles in the subatomic scale can be physically exchanged, and it is a conceptual problem that remains partially unsolved. For example, imagine that two particles are far away from each other, so far that they can not be the exchanged. Physically, the process in order to exchange the particles will never be complete. Other problem that arise is the fact that if the particles are identical, then an exchange makes no sense since the result will be the same. This is also named indistinguishability, and it will be important in the discussion of the anyonic system. Instead we would refer to exchange as the exchange of the particle labels and suppose that they must at least mathematically follow a path leading to that exchange. Suppose that the i-th particle is identified by the point $q_i \in \mathcal{Q}$(configuration space), then the exchange means to change for example $i \leftrightarrow j$ in the label of the particle and to assume that the particles follow a path in order to make this exchange. When one treats the exchange in this way, one can think of it as if there where a permutation in the particle's indexes. During symmetry exchange, a statistical

phase appears in the wave function, a $+1$ is associated with a symmetric wave function, while a -1 is associated with an anti-symmetric wave function (Salinas, 2013, p. 141). A symmetric wave function acquires a statistical phase of $+1$ hence the particles are bosons and they obey Bose-Einstein statistics. If the wave function is anti-symmetric the statistical phase is -1 then the particles are fermions and they obey Fermi-Dirac statistics (Theorem 1.1).

As have been said before the observables preserve the symmetric properties of the wave function and then this discussion will be focused on the wave function. The symmetry exchange is implemented in the following way

$$\Psi(q_1, q_2, ..., q_i, ..., q_j, ..., q_n) = \pm\Psi(q_1, q_2, ..., q_j, ..., q_i, ..., q_n). \tag{1.2}$$

The outlook up to now is what was done until 1950, it seemed that the quantum mechanical description of a system of identical particles was complete but this will result in a false assert. It turned out that quantum mechanics has some ambiguities present in its formulation, and also the dimension of the system matters in the description of a system of particles. Two dimensional systems ($d = 2$) named *Low Dimensional Systems* behave in a very different way, and the quantum description for this kind of systems changes dramatically, in part, due to the configuration spaces for this systems, and resulting topology. In the path integral formulation of the quantum theory, ambiguities appear by considering paths belonging to different homotopy classes. That induce different phase changes in the integrand under the same gauge transformations of the Lagrangian. This new feature of the path integral for this kind of systems of identical particles is named multiply-connected spaces, and it was first pointed out(in physics) by Schulman (1968) and later, a more general treatment was presented by Laidlaw and DeWitt (1971). They showed how the non-trivial topology of the configuration space leads to different inequivalent quantizations of the same classical system, which in turn affects the phase acquired by a wave function on a symmetry exchange.

This was the first time when the topology of the configuration space played an important role on the quantum theory of indistinguishable particles, and it proved to be something very strange because by treating low dimensional systems of identical particles ($d = 2$) a global phase arise, and it was different from $+1$ or -1 as it would be expected. A new and more general treatment of the configuration spaces on systems of indistinguishable particles was done by Leinaas and Myrheim (1977). Their approach was based on the geometrical interpretation of wave functions, which is the basis of the modern day gauge theories. They showed that in two dimensions, the space is multiply-connected which results in the possibility of what they termed as the intermediate statistics. In particular, they showed that there exists a continuously

variable parameter, which one can choose to be any phase angle θ, for which on one minimum value it recovered the bosonic statistics (phase $+1$) and for one maximum value it recovered the fermionic statistics (phase -1), suggesting that this statistical phase was intermediate between the both known, and the phase angle should have values on a certain range. This idea of intermediate statistics gained special attention with the papers published by Wilzeck, and it was he who coined the famous name Anyons[1] (Wilczek, 1982b) and anyonic statistics (also known as fractional statistics). He discovered a way to dynamically treat this kind of systems, this formulation was known as the flux tube model for anyons and consisted of assigning to a particle a delta flux function which will lead to a gauge transformation in the Lagrangian. With this description of an anyonic system, it was discovered that treating any anyonic system would be difficult since the particles are interacting. It is still a very difficult problem to solve the Schrödinger equation for a system with more than two particles, in fact computational methods are used to solve this kind of equations.

By 1983 this type of statistics gained popularity. This was due to the discovery of the Quantum Hall Effect(**QHE**) and the Fractional Quantum Hall Effect(**FQHE**). For the moment it is not important to explain it, so the explanation of the **QHE** will be made later with the concise discussion of the anyonic term in the hall effect. The first attempt to explain the **FQHE** was done by Laughlin. He proposed that the fractional quantization of the Hall resistance is the manifestation of a new state of matter, the incompressible quantum fluid, with elementary excitations which have fractional charge (R. B. Laughlin, 1983). Also this kind of particles were treated as bosons and that lead to some problems. The anyonic interpretation which was successful in the explanation of the **FQHE** was given by Halperin. He suggested that quasiparticles at the fractional quantized Hall states obey quantization rules appropriate to particles of fractional statistics (Halperin, 1984). Since then, the anyonic statistics offered a variety of different conceptions of applications in other branches of physics, like the high temperature anyonic superconductors (Chen, Wilczek, Witten, & Halperin, 1989; R. B. Laughlin, 1988; Wilczek, 1990), or the realization of canonical ensemble and the virial coefficients in anyonic statistics (D. P. Arovas, Schrieffer, Wilczek, & Zee, 1985; Bhaduri, Bhalerao, Khare, Law, & Murthy, 1991; Comtet, Georgelin, & Ouvry, 1989; Dowker, 1985; Lerda, 2008), also they are conceptually proposed as a solution to quantum computing (Averin & Goldman, 2001; Freedman, Kitaev, Larsen, & Wang, 2003; Khare, 2005; Kitaev, 2003; Mochon, 2003; Williams, 1999).

This work deal mainly with the study of an anyonic system. First by the inspection of the configuration space of a system of indistinguishable particles, in order to understand how mathematically a low dimensional system can produce some important results on a wave function

[1] This arose from the famous tradition of puting an "on" to the discovered particles, and the fact that the anyon can be "any" type of particle in between bosons and fermions so this particles were named "any-ons"

describing this system. The correct use of the important results obtained will be fundamental in the physical description of an anyonic system. This all due to the use of braids and braid diagrams. Later on a complete dynamical analysis, those mathematical properties will be understood. This by using the flux tube model of anyons, and the Schrödinger picture of a two particles low dimensional system. First with an harmonic potential, and later on with an uniform magnetic field potential. The last result will be important since the main application of the anyonic formalism is the FQHE. This effect will be understood first by quantizing a system under an uniform magnetic field, and later by introducing the anyonic formalism.

Chapter 2

Mathematical Description of an Anyonic System

In this chapter the main description of identical particles will be developed. First of all, it will be shown how quantum mechanics has some ambiguities in its description. After seeing that, a system of indistinguishable particles will be studied and from the configuration space, some relevant features will be developed. Specially with systems composed of $N = 2$ particles. The next part of the discussion, will deal mainly with the mathematical structure of this kind of systems for the case of low dimensional systems and higher dimensional systems. This by considering a symmetry exchange, after the study of the configurations space. The consideration of the fundamental group will be understood by realizing how does the fundamental group for two dimensional systems differ from the usual higher dimensional systems. To end this chapter the use of the quantization by path integrals will lead to the one dimensional representations of the braid group.

2.1 Ambiguities in Quantum Mechanics

According with the description made by Reyes (2006), quantum mechanics possesses certain ambiguities because what matters physically is the amplitude of the wave function $\|\psi(q)\|^2$. To see this, let us consider the mathematical underpinnings of the Schrödinger quantization and the Feynman quantization. In quantum mechanics states of particles are either pure states or mixed states (Scheck, 2007, p. 200). A pure state can be described by a single ket while a mixed state is a statistical ensemble of pure states. In the Hilbert space a pure state can be treated as

a ray due to the gauge freedom of the wave-function. To see this clearly, let us consider a wave function $\psi : \mathcal{Q} \to \mathbb{C}$ where $\mathcal{Q}$ is the classical configuration space. Now what physically matters is the probability amplitude, that is $\|\psi(q)\|^2$. We have the freedom to add a term to this wave function such that the probability amplitude remains the same. This term is a global phase $e^{i\theta}$. Indeed this phase can be generalized to a local one like $e^{i\phi(q)}$, such that the wave-function takes the form $\psi(q) \mapsto e^{i\phi(q)}\psi(q)$.

The same occurs in the path integral formulation. As it is well known, we can add a term to the Lagrangian such that the Euler-Lagrange equations remain invariant(Fasano & Marmi, 2006, p. 126). That term must have the form $d\left(A(q,t)\right)/dt$. Consider a classical system defined on a configuration space $\mathcal{Q}$ described by a Lagrangian $L(q,\dot{q};t)$. The gauge transformation defined on this configuration space is

$$L(q,\dot{q};t) \longrightarrow L'(q,\dot{q};t) = L(q,\dot{q};t) + \frac{\mathrm{d}}{\mathrm{d}t}\left(A(q,t)\right). \tag{2.1}$$

In the path integral formulation, the probability $P(b,a)$ to go from a point x_a at time t_a to the point x_b at time t_b is the norm squared $P(b,a) = \|K(q_b,t_b;q_a,t_a)\|^2$ of an amplitude $K(q_b,t_b;q_a,t_a)$ to go from a to b. This amplitude is defined as the sum of contributions from each path, and it has a phase proportional to the action S. It is written in general as (Feynman, Hibbs, & Styer, 2010, p. 35)

$$K(q_b,t_b;q_a,t_a) = \int e^{\frac{i}{\hbar}S[q(t)]}\mathcal{D}[q(t)]. \tag{2.2}$$

It can be seen that the overall change in the amplitude of Equation 2.2 is a global phase $\phi(q_b,t_b;q_a,t_a)$,

$$K(q_b,t_b;q_a,t_a) \longrightarrow K'(q_b,t_b;q_a,t_a) = e^{\frac{i}{\hbar}\phi(q_b,t_b;q_a,t_a)}K(q_b,t_b;q_a,t_a), \tag{2.3}$$

where $\phi(q_b,t_b;q_a,t_a) = A(q_b,t_b) - A(q_a,t_a)$. This implies that the phase only has a dependence on the extremal points, which are held fixed. This situation changes drastically when the configuration space has "holes". This can be seen with an example. Consider the movement of a free particle on a plane with a "hole" on it

$$\mathcal{Q} : \mathbb{R}^2 \setminus \{0\}\,; \quad L(q,\dot{q};t) = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2). \tag{2.4}$$

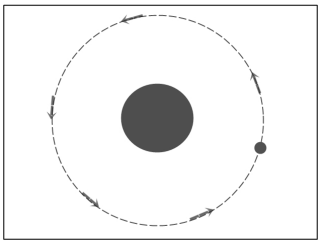


FIGURE 2.1: Free particle on a plane with a "hole" on it

Now consider the function

$$\begin{array}{rcl} F : T\mathcal{Q} & \longrightarrow & \mathbb{R}^2 \\ q & \longmapsto & (F(q_1), F(q_2)) = \frac{1}{2}\left(\frac{-q_2}{q_1^2+q_2^2}, \frac{q_1}{q_1^2+q_2^2}\right), \end{array} \tag{2.5}$$

this function describes the movement of the particle on a plane in a circular path. Notice that $\partial_2 F_1 = \partial_1 F_2$, since

$$\partial_2 F_1 = (q_2^2 - q_1^2)/(q_1^2 + q_2^2) = \partial_1 F_2. \tag{2.6}$$

This implies that

$$(F(q_1), F(q_2)) = \left(\frac{\partial A}{\partial q_1}, \frac{\partial A}{\partial q_2}\right), \tag{2.7}$$

using a gauge transformation, the Lagrangian can be written as

$$L'(q, \dot{q}; t) = L(q, \dot{q}; t) + F_1 \dot{q}_1 + F_2 \dot{q}_2. \tag{2.8}$$

Now introducing Equation 2.8 into the action integral, it can be seen that there is a phase in the path integral like in Equation 2.3. This phase is somewhat different

$$S'[q(t)] = \int \left(L(q, \dot{q}; t) + \frac{1}{2}\left(\frac{q_1 \dot{q}_2}{q_1^2 + q_2^2} - \frac{q_2 \dot{q}_1}{q_1^2 + q_2^2}\right)\right) dt, \tag{2.9}$$

$$S'[q(t)] = S[q(t)] + \phi[q(t)]. \tag{2.10}$$

Now let us consider the family of paths of the form $\gamma_n(t) = (\cos(2n\pi t), \sin(2n\pi t)) = (q_1(t), q_2(t))$ with $n \in \mathbb{Z}$. This family of paths represent a circular movement on the plane as in the example. Nevertheless, the paths belonging to different values of n are not deformable into each other, which is a really important property. Using the relation above and $q_a = 0 = q_b$, $t_a = 0, t_b = 1$,

$$\phi[q(t)] = \int_0^1 \left(\frac{1}{2} \left(\frac{q_1 \dot{q}_2}{q_1^2 + q_2^2} - \frac{q_2 \dot{q}_1}{q_1^2 + q_2^2} \right) \right) dt, \tag{2.11}$$

$$\phi[q(t)] = \int_0^1 n\pi(\cos(2n\pi t)^2 + \sin(2n\pi t)^2)dt = n\pi, \tag{2.12}$$

and by Equation 2.3

$$K'(q_b, t_b; q_a, t_a) = e^{\frac{i}{\hbar} n\pi} K(q_b, t_b; q_a, t_a). \tag{2.13}$$

In both considerations for quantization the same kind of phase appeared $e^{i\theta}$. This ambiguities considered lead to the main feature of an anyonic system. That is, a continuous phase that can lead to any value in between both statistics. To see this we shall consider a system of identical particles.

2.2 Configuration space for a system of indistinguishable particles

Dealing with a system of particles is a hard problem(Gibbs, 1878). A form to describe it correctly is to assume that the particles composing this system are all identical and indistinguishable. A system of particles is described by a wave function like Equation 1.1, and assuming indistinguishability one cannot make a distinction between particles. In fact, the description of this kind of system lead us to know about the statistics that a system obeys. The easiest way to begin is to consider a system of particles that first of all are indistinguishable and secondly, every particle on this system must be hard. That means that two particles can not be in the same position.

Following Leinaas and Myrheim (1977), let us consider now a system of identical particles. Here we will deal with the configuration space for N particles. Let us define $\mathcal{Q}$ as one particle configuration space. With this in mind we might be tempted to state that the N particles configuration space is just $\mathcal{Q}^N$. Although this is true locally, but globally is not. Indeed,

globally any permutation of the particles would lead to the same physical configuration, since particles are indistinguishable.

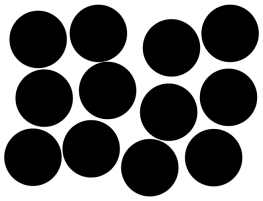


FIGURE 2.2: Local scheme of $\mathcal{Q}^N$

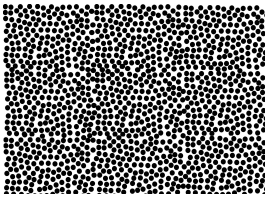


FIGURE 2.3: Global scheme of $\mathcal{Q}^N$

When one talks about an exchange of particles, we understand it as a permutation on the points in the configuration space representing the state of the particle[1]. Suppose that we have a state defined by $q = q(q_1, q_2, \ldots, q_N) \in \mathcal{Q}^N$ with $q_i \in \mathcal{Q}$. When an exchange is done the resultant state is a permutation of the original point, differing only in the labeling $q' = Pq = q'(q_{p(1)}, q_{p(2)}, \ldots, q_{p(N)})$. Since particles are indistinguishable, there will not be any difference between the original point and the exchanged point. Therefore, the configuration space is not $\mathcal{Q}^N$ but is the space which is obtained by identifying points in $\mathcal{Q}^N$, that represents the same physical configuration (they differ only in the ordering). To identify points one then uses the action of the symmetric group on a given set. Thus, the configuration space is the set taking out all possible configurations of $\mathcal{Q}^N$ by using S_N. Here S_N is a finite group, and for this problem it represents the action on the configuration space.

The configuration space treated up to now needs to be more concise since physically the particles are hard, and they must be identified for the "exchange". For N particles the general configuration space is as follows. We will introduce the condition of hardness by introducing what is known as the generalized diagonal. This is $\Delta = \{q_1, ..., q_N \in \mathcal{Q}^N | q_i = q_j \text{ for some } i \neq j\}$. Now, the configuration space is the one in which it is not possible for two particles to be in the same "position[2]" at the same time, so two particles cannot coincide. Also the points in the configuration space need to be labeled since it is fundamental to identify configurations that differ only in the ordering. Since there are N particles, the configuration space then will be $\mathcal{Q}^N = (\mathbb{R}^d)^N - \Delta$, with $\mathbb{R}^{Nd}$ being the N particles d-dimensional euclidean space. Taking out configurations that differ in the ordering the general configuration space is written as

$$\frac{\mathcal{Q}^N}{S_N} = \frac{(\mathbb{R}^d)^N - \Delta}{S_N}. \tag{2.14}$$

[1] They must at least mathematically follow a path in order to exchange their positions

[2] Since we are interested in the quantum viewpoint, it is important to say that we must take out the probability of particles to be in the same position

In studying a system of particles it is necessary to introduce the center of mass coordinate. The mathematical space considered is thus that corresponding to the center of mass. It is important to say that is an d-dimensional euclidean space. Even though we are dealing with permutation of the indices labeling the particles, then the center of mass coordinate will be invariant under the action of S_N [3]

$$\mathbf{R} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}_i \in \mathbb{R}^d. \tag{2.15}$$

To complete the description, a new space will be introduced. That is, the space corresponding the relative motion between particles. In general it will be written as $r(d, N)$, where d gives account for the dimension of the system and N for the number of particles. This space contains the property of hardness and is invariant under the identification of the relative motion. So, the general configuration space is thus that one corresponding to the center of mass and the relative motion. Written as,

$$\frac{\mathcal{Q}^N}{S_N} = \mathbb{R}^d \times r(d, N). \tag{2.16}$$

Low dimensional systems are those in two space dimensions $d = 2$. An anyon is a low dimensional particle, that is, a particle in two space dimensions. We might wonder about the difference between studying systems in $d \geq 3$, and studying systems in $d = 2$. In order to understand this we will only treat systems of $N = 2$ for simplicity and convenience for what follows. Consider two free particles in $\mathbb{R}^d$ like in Figure 2.4[4]

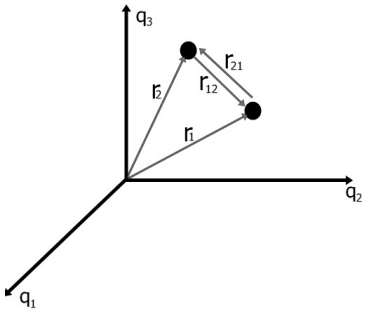


FIGURE 2.4: Two free particles in the Euclidean space $\mathbb{R}^d$

As mentioned before the coordinates for this system are that corresponding to the center of mass and the relative motion,

[3] Boldface will be used to denote vectors

[4] for obvious reasons there is no better picture for $\mathbb{R}^d$

$$\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) \in \mathbb{R}^d, \tag{2.17}$$

$$\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2 = -\mathbf{r}_{21} \in \frac{\mathbb{R}^d - \{0\}}{S_2}. \tag{2.18}$$

The last equation holds since two particles cannot be in the same "position". Then it is mandatory to take out the point where two particles can coincide, that point is $\mathbf{r}_{12} = 0$. It is also necessary to take $\mathbf{r}_{12} = -\mathbf{r}_{21}$, since the particles are identical and the two points describe different ordering differing by a sign, just like an even or an odd permutation. Thus the configuration space associated with this system will be

$$\frac{\mathcal{Q}^2}{S_2} = \mathbb{R}^d \times \frac{\mathbb{R}^d - \{0\}}{S_2} = \frac{(\mathbb{R}^d)^2 - \{0\}}{S_2}. \tag{2.19}$$

Equation 2.19 is the configuration space for a two particle system. The differences between low dimensional systems $d = 2$ and systems in $d \geq 3$ will be clear by studying the relative configuration space $r(d, 2)$.

2.2.1 Systems in higher dimensions $d \geq 3$

Since points representing states differ by a permutation of the indices labeling the particles, when on deals with the overall configuration space on must take out all possible permutations that can represent the same physical state. To imagine this, assume for example two particles that have a labeling a,b. One can have two possible states in the configuration space which are ab or ba since particles are indistinguishable. If one takes out that two possibilities, one would have instead that there are two particles independent of their ordering, which is the result of being physically indistinguishable.

The relative configuration space can be identified as a "surface" of a $d-1$ dimensional sphere S^{d-1} with the diametrically opposite points identified (Comtet, Jolicoeur, Ouvry, & David, 1999, p. 280) by $\mathbf{r}_{12} = -\mathbf{r}_{21}$. By definition this is the real projective space P_{d-1}(Gallier, 2011, p. 107).

$$P_{d-1} = \frac{\mathbb{R}^d - \{0\}}{\sim}. \tag{2.20}$$

In order to exchange their positions the particles must follow continuous points on the relative configuration space, so that we are dealing with a path taken in order to make the desired exchange. By taking Equation 2.20 we can consider three kinds of possible paths on $r(d, 2)$.

1. **No exchange path:** In this case particles move, but there wont be any exchange between them. So, in the real projective space the path identified by η_1 is a closed path, since there is not any exchange (a loop).

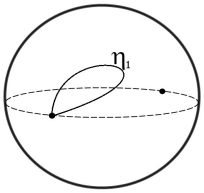


FIGURE 2.5: No exchange path η_1 between the particles

   This path possesses an important topological property, namely it is simply connected. That is, it can be continuously deformed by into a point (Munkres, 2000, p. 333). This is an important property, since it means that this kind of paths cannot impart any kind of phase into the wave function. It is pretty obvious since there is not any exchange. We can relate any simply connected space to the fact that they cannot impart any phase to the wave function under this considerations.

2. **Single exchange path:** In this case there is an exchange between the particles. So, in the real projective space the path η_2 from any initial point to its diametrically opposed is a closed path.

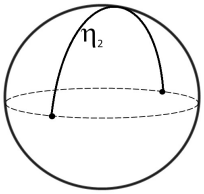


FIGURE 2.6: Single exchange path η_2 between the particles

   This path is not simply connected since it cannot be continuously deformed to a point, even though it is a closed path (diametrically opposed points are identified). Let us suppose that this path cause non-trivial and unknown phase into the wave function.

3. **Two exchanges path:** In this case there are two continuous exchanges between particles. So, the path is a concatenation of two single exchange paths. In the real projective space the path η_3 is a closed path.

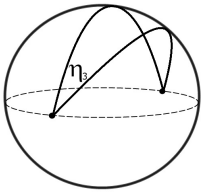


FIGURE 2.7: Two continuous exchanges path η_3 between the particles

This path is again simply connected since two consecutive exchanges $\eta_2 * \eta_2 = \eta_3 = \eta_1$ means no exchange at all and so the path taken can be continuously deformed into a point.

2.2.2 Systems in dimension $d = 2$

In this case the relative configuration space $r(2, 2)$ is associated with a plane with a hole having the opposite points being identified. This can be thought as a circle S^1 by making a single retraction of the full space. Let us study the same paths studied before to the $d \geq 3$ case.

1. **No exchange path:** In this configuration, there will not be any exchange between the particles. So the path γ_1 in the relative configuration space is a line on S^1

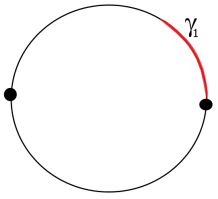


FIGURE 2.8: No exchange path γ_1 between the particles

Making an analogy with the case treated before this path can be continuously deformed into a point in S^1(is simply connected). Like before it is an important property since it means that in low dimensional systems this kind of paths cannot impart any kind of phase into the wave function.

2. **Single exchange path:** In this case there is only one exchange between the particles. In S^1 the path γ_2 is a closed path by the definition of the configuration space.

   This path is not simply connected even though it is a closed path. Then, let us suppose that this kind of paths induce an non-trivial unknown phase in the wave function under a symmetry exchange.

3. **Two exchanges path:** In this case there are two continuous exchanges between the particles. In S^1 the path γ_3 to make the exchange is a closed path.

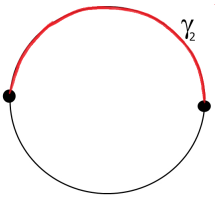


FIGURE 2.9: Single exchange path γ_2 between the particles

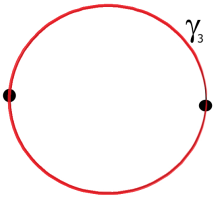


FIGURE 2.10: Two continuous exchanges path γ_3 between the particles

This path γ_3 is very different from the path η_3 on $d \geq 3$, because in this case it is not possible to continuously deform the path into a point in the circle. Then the space is not simply connected.

In two dimensions $d = 2$ three kind of closed paths exist. They correspond to no exchange γ_1, one exchange γ_2, and two consecutive exchanges γ_3. The paths on $d = 2$ differ from those on $d \geq 3$. In the first case three kind of closed paths exists, while in the latter case two kind of closed paths exist, but this differentiation seems trivial. So, the main properties of these systems may be treated with the use of a classification on spaces. This is done by founding the fundamental group of the aforementioned configuration spaces.

When one talk here of paths, that means paths η , γ defined on the above said topological spaces. As the main interest here are the closed paths, we will see how this paths relate into each other. To do this let us introduce a equivalence relation on this topological spaces, namely "to be homotopy equivalent", in layman words that means that we can take any path e.g α and by a continuous mapping H transform it into any other say β. As it is well known equivalence relations can define partitions on a set by taking the equivalence classes of the set. Here all the paths that are homotopic form equivalence classes $[\alpha]$ that will be named homotopy classes, this homotopy classes define a group under the concatenation of paths. The group of all the homotopy classes of closed paths is named the fundamental group of the space, and it is written in general as $\pi_1(X)$ where X is the topological space.

2.2.3 The Fundamental Group for $\frac{\mathcal{Q}^N}{S_N}$

One may wonder about the importance on the fundamental group in this problem of anyons. It happens that the fundamental group gives us information about the type of space, if any path α on a topological space X can be srhunk into any point $x_0 \in X$ (a contraction) this space is named simply connected, otherwise it is named multiply-connected. Schulman (1968) pointed out that by studying the path integral for the spin, one may see how the connectedness of the space lead us to different inequivalent quantizations if the space results to be multiply-connected. So, the connectedness of the space is crucial in the quantizacion process. Even as the connectedness is defined by the fundamental group, the latter plays a core role on the study of the low dimensional systems, therefore it has to be found. The problem of finding homotopy groups is a hard problem even for the most simple topological spaces, and for the sake of the anyonic statistics it happened that the fundamental groups concerned to topological spaces like in Equation 2.19 had already been found by Fadell and Neuwirth (1962); Fadell, Van Buskirk, et al. (1962); Fox and Neuwirth (1962). It resulted that the fundamental group for the configuration space of the discussed system is

$$\pi_1\left(\frac{\mathcal{Q}^N}{S_N}\right) = \begin{cases} B_N & \text{if} \quad d = 2 \\ S_N & \text{if} \quad d \geq 3 \end{cases} \tag{2.21}$$

Equation 2.21 tell us how the fundamental group changes if different dimensions are treated in a system where there are N particles, this is the most general result. Of course for the sake of simplicity and convenience setting $N = 2$ would lead us to some more specific result which can be interpreted in an easier way. Then the fundamental group for our specific problem becomes

$$\pi_1\left(\frac{\mathcal{Q}^2}{S_2}\right) = \begin{cases} B_2 & \text{if} \quad d = 2 \\ S_2 & \text{if} \quad d \geq 3 \end{cases} \tag{2.22}$$

Equation 2.22 show clearly the main differences between $d \geq 3$ and $d = 2$. In the three dimensional case the fundamental group is S_2, this group tell us nothing about the path that the particles took to make an exchange, since it deals with the all the functions that lead to a permutation on a given set. If we take a permutation in a two elements set, two consecutive permutations lead to the initial configuration of the set. In two dimensional case, the fundamental group is B_2 which is known as the Braid Group. This group is very different from the symmetric group, since it physically can be related with the paths that the particles individually take, to give a meaning to the problem every particle can be treated as a strand in a braid diagram. To

see a more rigorous approach to the braid group see Appendix A. Here the main difference can be portrayed in the following way.

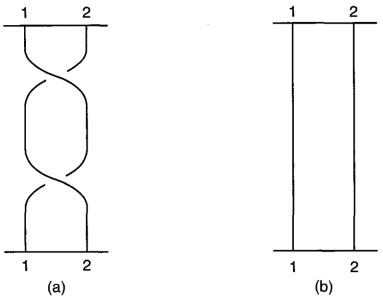


FIGURE 2.11: Double exchange of particles, a). in the braid group, b).in the symmetric group(Khare, 2005, p. 41)

As discussed before, in three dimensions two consecutive exchange means no exchange, since it is a simply connected space and it can not impart any phase to the wave function. To get a better picture of this difference, take for example Figure 2.11. In the first case, two continuous exchanges are the same as no exchange as portrayed in Figure 2.11 B). In the second case, where the fundamental group is the braid group how one particle wind around the other matters Figure 2.11 a), indeed also the angle is important and have a meaning. The main difference is thus that after two continuous exchanges particles in the braid group differ from the initial configuration, as there is a braid. It is hard to portray what a braid is but physically a meaning can be assigned. Every horizontal line represents the space, while a vertical line represents the evolution of the particle in time. According to the definition Every generator named σ_i means to wind counter-clockwise one particle around the other, with two particles one only has one generator(Definition A.1). Up to now the main descriptions of low dimensional systems have been treated in a simple way. Now the rigorous approach can lead us to the desired results in a useful way.

2.2.4 The path integral formulation for multiply connected spaces

Wave mechanics proposes a problem to all this definitions that have been discussed, since as it is well known in quantum mechanics one does not use the description of a system based on the path followed by the particle, since the Heisenberg inequalities lead to not knowing about the position of a particle. The solution to this setback is simple. One just use a different method for the quantization of the system. One method in which the path matters. As it is well known this method for quantization is the path integral formalism. In this formalism one just describe the amplitude of a wave function by the integration over all possible paths in which a system

can evolve, and it is defined as Equation 2.2. Then, a rigorous treatment of this problem may be solved using the path integral formalism.

According to Schulman (1968), if a space happens to be multiply connected, paths defined on that spaces would lead to a multivalued wavefunction which in turn lead to a multiple quantizations of the same system. This means, that for example if two paths $\gamma_1, \gamma_2 \in \pi_1(\frac{\mathcal{Q}^N}{S_N})$ belonging to different homotopy classes are considered, the overall probability amplitude would be ambiguous, since as all paths are considered, one class of paths can induce a variation in the amplitude. The proposed solution is just to instead of taking all possible paths, take a weighted sum of homotopy classes belonging to the fundamental group. And instead of taking a total amplitude take partial amplitudes associated with the homotopy classes

$$K = \sum_{\alpha \in \pi_1(\frac{\mathcal{Q}^N}{S_N})} \chi(\alpha) K^\alpha. \tag{2.23}$$

According to Laidlaw and DeWitt (1971), the weight factors $\chi(\alpha)$ must form a one dimensional representation of the fundamental group. Their proof is general and contains the results obtained by schulman. It can be sketched in the following way, let x be some fixed point of a topological space X and let $\alpha, \beta, \ldots, \eta \in \pi_1(X, x)$. Let $C(a)$ be any path connecting a to x where $a \in X$ and can be arbitrary. Then we can be define the *Homotopy mesh* as a mapping from the homotopy group with a base point and the homotopy clases with initial and final points defined in the following way.

$$\begin{gathered} f_{ab} : \pi_1(X, x) \longrightarrow \pi_1(X, a, b) \\ f_{ab}(\alpha) = [C(a)^{-1}]\alpha[C(a)] = K^\alpha(b, t_b; a, t_a) \end{gathered} \tag{2.24}$$

This mesh preserves the operation in the homotopy classes, which forms the homotopy group $f_{ab}(\alpha) f_{bc}(\beta) = f_{ac}(\alpha\beta)$. The meaning of the homotopy mesh, is to associate to every homotopy class of paths between points $a, b \in X$ an element of the fundamental group. In order to relate a path to an element of the fundamental group. It is important to mention that this association is not arbitrary, and neither is unique since it depends on the choice of the mesh.

The dynamics of the system will be given by a lagrangian in X, then the partial probability amplitude let us say K^α will be obtained by performing the path integration over all paths in the homotopy class $f_{ab}(\alpha)$. Laidlaw and DeWitt Proposed four properties for the partial amplitudes to satisfy, they are:

- The composition of amplitudes is given in general by

$$K^{\gamma}(c,t_c;a,t_a) = \sum_{\alpha\beta=\gamma\in\pi_1(X)} \int db\; K^{\beta}(c,t_c;b,t_b)K^{\alpha}(b,t_b;a,t_a). \tag{2.25}$$

- For every point $a \in X$ there exists an open set $u \subset X$ containing a such that if $a' \in u$, then $K^{\gamma}(a',t_{a'};a,t_a) \to 0$ as $t' \to t, (t' \neq t)$ for all but one and only one homotopy class.
- If there is a change in the homotopy mesh $C \to \bar{C}$ there will be a change in the partial probability amplitude $K^{\alpha} \to \bar{K}^{\alpha} = K^{\lambda\alpha\mu}$ where $\lambda, \mu \in \pi_1(X)$.
- Linear independence.

In order that the total probability amplitude propagate the states of the system, it is necessary that the overall amplitude fulfill the quantum rule for probabilities. For any concatenation of paths which lead to composition of amplitudes one has (Feynman et al., 2010)

$$|K(c,t_c;a,t_a)| = \left| \int db\; K(c,t_c;b,t_b)K(b,t_b;a,t_a) \right|. \tag{2.26}$$

When supposing multiply connected spaces, one then has to use the first property in order to establish a correct rule for probabilities

$$e^{i\varepsilon}K(c,t_c;a,t_a) = \sum_{\alpha,\beta\in\pi_1(X)} \chi(\alpha)\chi(\beta) \int db\; K^{\beta}(c,t_c;b,t_b)K^{\alpha}(b,t_b;a,t_a). \tag{2.27}$$

The phase factor $e^{i\varepsilon}$ arises from the general law for probabilities, since it does not change the overall probability. Changing the mesh according to the third property, to all points $b \in X$ except for $b \neq a, c \in X$ such that $\bar{f}_{ab}(\alpha) = f_{ab}(\alpha\gamma)$ and $\bar{f}_{bc}(\beta) = f_{bc}(\gamma^{-1}\beta)$, and using the operation rule for the mesh one obtains

$$e^{i\varepsilon} \sum_{\alpha\beta=\delta\in\pi_1(X)} \chi(\delta)K^{\delta}(c,t_c;a,t_a) = \sum_{\alpha,\beta\in\pi_1(X)} \chi(\alpha)\chi(\beta)K^{\alpha\beta}(c,t_c;a,t_a). \tag{2.28}$$

As established before, for concatenation of paths and composition of amplitudes one has $\alpha\beta = \delta$. Using the last property and Equation 2.25 the final result is

$$e^{i\varepsilon}\chi(\delta) = \chi(\alpha)\chi(\beta). \tag{2.29}$$

Here ε is arbitrary and is general, so taking $\varepsilon = 0$ will not affect the overall probability amplitude. Thus Equation 2.29 can be written as

$$\chi(\alpha\beta) = \chi(\alpha)\chi(\beta). \tag{2.30}$$

Equation 2.30 Is the definition of a one dimensional representation (Tung, 1985, P. 28). In this case a one dimensional representation for the fundamental group, considering a space which is multiply connected. This last result allow us to search for a one dimensional representation for our specific problem, in order to find a way to understand exchanges. It happens that for the case of $d \geq 3$ this problem was also solved by Laidlaw and DeWitt. According to Equation 2.21 for $d \geq 3$ the fundamental group is the symmetric group S_N. The only one dimensional representations for the symmetric group are

- $\chi^1(\alpha) = 1$ for all α,
- $\chi^2(\alpha) = \pm 1$ according to the character of α to be an even permutation or to be an odd permutation.

It is important to understand that $\alpha \in \pi_1(\frac{\mathcal{Q}^N}{S_N})$ represents a path, then the odd character or the even character of the permutation means to associate a path to the statistics that a system obeys. Thus, in $d \geq 3$ the propagators become what it is well known under a symmetry exchange, that is

$$K^{\text{Bose}} = \sum_{\alpha} \chi^1(\alpha) K^{\alpha}, \tag{2.31}$$

$$K^{\text{Fermi}} = \sum_{\alpha} \chi^2(\alpha) K^{\alpha}. \tag{2.32}$$

For the the case of low dimensional systems ($d = 2$). The problem was still very hard to solve, since it was still difficult at that time to find one dimensional representations of the braid group. That problem lasted unsolved for about 13 years and then it was solved by using the same type of arguments, but with a physical insight that lead to the correct solution. The solution was proposed by Wu (1984a). In general, the solution to find a one dimensional representation of the braid group (low dimensional systems) had to agree with the results obtained by Laidlaw and DeWitt (1971), and to recover the general quantum rule for probabilities.

According to Appendix A braids are represented as diagrams in which the horizontal line represents the two dimensional space for our problem and the vertical line represents a parametric evolution of the base points. As can be seen better the braid group in general is an abelian group, so by Lemma A.7 any one dimensional homomorphism is cyclic. which means that indeed one can find a one dimensional representation that fulfill Lemma A.7. The only property that this one dimensional representation need to fulfill is $\chi_\theta(\sigma_1) = \chi_\theta(\sigma_2) = \cdots = \chi_\theta(\sigma_{N-1})$. Note that a common value can be given to this $\chi_\theta(\sigma)$ in order to recover the property. The chosen form that this common value has is customary to choose, and proposed ny Wu to be

$$\chi_\theta(\sigma) = e^{i\theta}, \tag{2.33}$$

where $0 \leq \theta < 2\pi$. In order to achieve the property that one can wind one particle around the other, clockwise or counter-clockwise. We will choose to take this phase in the following way

$$\chi_\theta(\sigma) = e^{\pm i\theta} = \exp\left\{ i\frac{\theta}{\pi} \sum_{i<j} \Delta\phi_{ij} \right\}. \tag{2.34}$$

The new term into the exponential $\Delta\phi_{ij}$ is the azimuth angle, it can be positive or negative and it gives account for the rotation of one particle around the other. The sub indices label the particles and the sum last for the total system. In this particular case where there are just two particles there will only be one azimuth angle. A way to introduce this into the path integral amplitude, is to make this azimuth angle a continuous function of time. Of course this assumption is taken in order to understand the winding of particles physically. It will be important to parametrize this angle in terms of the time. Thus, without loss of generality the phase can be written as

$$\chi_\theta(\sigma) = e^{\pm i\theta} = \exp\left\{ i\frac{\theta}{\pi} \int dt \frac{d}{dt} \sum_{i<j} \phi_{ij}(t) \right\}. \tag{2.35}$$

By retrieving from Equation 2.23 the overall change in the partial amplitudes. And by using the aforementioned lemma, the amplitude will be a weighted contribution, but as the braid group is cyclic under one dimensional representations (abelian nature and only just one generator of the braid group for $N = 2$) one then has, that the full contribution is just the general common term in Equation 2.34. Also by requirement we have $q_b = q = q_a$, so we can treat fully the problem of the exchange of particles by considering closed paths. Therefore

$$K(q,t_b;q,t_a)=\int \mathcal{D}[q]\exp\left\{\frac{i}{\hbar}\int dt\left\{L+\frac{\theta}{\pi}\frac{d}{dt}\sum_{i<j}\phi_{ij}(t)\right\}\right\}. \tag{2.36}$$

From the last equation one can see that the overall change in the amplitude is indeed caused by a general phase which was found to be Equation 2.35. But also from Equation 2.36 one can point out a really important conclusion. In fact this conclusion lead to the full description of anyons dynamically. That is, the adding of a phase that fulfill the mathematical requirements up to now add a term into the action. This term includes a quantity which is a function of time. As stated before this is a gauge transformation of the lagrangian. Then, this means first of all that the equations of motion are invariant under this transformations, so particles obeying this gauge transformation are treated as free particles. The second conclusion is that the overall phase acquired by the wave function of two particles under a symmetry exchange is completely different from the usual according to Theorem 1.1, then this particles are neither bosons nor fermions.

Up to now what was made, was to develop the main formalism of the anyonic statistics. And all the mathematical intuition that it carries. But this is not so useful, since anyons are still particles with unknown physical properties, little information is given by the fundamental group about the mass, the charge, maybe the spin[5]. So, what is left is to study the anyons in a dynamical way that agrees with the mathematical results obtained.

[5] A central issue in the anyonic statistics

Chapter 3

Physical Description of an Anyonic System

In the later chapter the overall phase that characterizes an anyonic system was found. Using the topological property of the configuration space of being multiply-connected, this property was exploited in order to find the fundamental group of this configuration space, leading to the braid group. Then, the overall phase was found to be $e^{i\theta}$ and it came from the one dimensional representation of the braid group for two particles. In this chapter the dynamical properties of anyons will be studied in order to obtain a physical representation of anyons. All this in order to understand the basic quantum mechanics of anyons(anyonic statistics or fractional statistics), and to solve the Schrödinger equation for a low dimensional system composed with two particles.

3.0.1 Aharonov-Bohm effect and the phase shift

In order to give a dynamical explanation of an anyonic system (i.e. that particles acquire a statistical phase when exchanged in two dimensions), Wilczek (1982a) used the fact that charged particles following a path in presence of a solenoid acquire a phase in the wave function due to the electromagnetic potential, even though no electric or magnetic field is there. This is named the Aharonov-Bohm effect(Aharonov & Bohm, 1959, 1961). This effect (the magnetic case) was described using a beam of electrons and a very long solenoid, with the property that no electron can get inside the solenoid, so the configuration space is a multiply-connected space.

It happens that the path that the electrons follow is affected by the magnetic potential, which induces a general phase shift on the wave function $\Psi : \mathcal{Q} \to \mathbb{C}$ by $\Psi(\mathbf{r}) \to \Psi'(\mathbf{r}) = e^{i\rho(\mathbf{r})}\Psi(\mathbf{r})$. This phase has the form

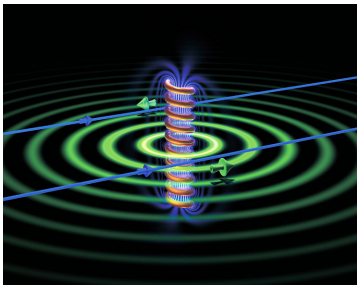


FIGURE 3.1: Magnetic Aharonov-Bohm effect on a solenoid (Batelaan & Tonomura, 2009)

$$\rho(\mathbf{r}) = \frac{q}{\hbar}\int_{\gamma} \mathbf{A}\cdot d\mathbf{r} = \frac{q\Phi}{2\hbar}. \tag{3.1}$$

For example, if two paths are taken so that the concatenation of both of them became a closed path, one has a phase shift for both paths. So that the general phase difference $\Delta\rho(\mathbf{r})$ is

$$\Delta\rho(\mathbf{r}) = \frac{q}{\hbar}\oint \mathbf{A}\cdot d\mathbf{r} = \frac{q\Phi}{\hbar}. \tag{3.2}$$

The phase shift induced by the magnetic potential resembles the anyonic phase acquired under a symmetry exchange, in two aspects. First, the phase is a pure quantum mechanical effect since there is no magnetic field in the path of the electrons. Secondly, the fact that the electrons can not be inside the solenoid makes the space multiply connected if concatenation of paths is made. In fact this was the main idea behind the dynamical formulation of an anyonic system. So, in order to describe an anyon one must at least generate a phase shift, and this can be done by using a solenoid. This proposes a difficult idealization of anyons since charged particles do not come around a solenoid so often in nature.

3.0.2 Flux tube model of Anyons

The aharonov-bohm effect proposes a form of adding a phase in a quantum mechanical way and practically solve the problem, but there is an issue with this. One must add a solenoid in order for the charged particles to acquire a phase in the wave function describing them. That means, that anyons must be charged particles. Other consideration is that even if the solenoid is added to the system, there is a low dimensional consideration which has not been studied. Then, a correct formulation of an anyonic system must have this properties on its dynamical description. The properties said before where considered in a model proposed by Wilczek (1982a, 1982b) and later presented in an simpler way by (Wilczek & Shapere, 1989, p 287.). This model started

by defining that an anyon particle is first of all a charged particle, second it is a particle that it is orbiting an infinitesimal solenoid that possesses a magnetic flux. This flux must be treated according the Aharonov-Bohm effect, so there must be a magnetic field that is zero everywhere except inside the solenoid. The way to implement this is to define a magnetic vector potential $\mathbf{a}(\mathbf{r})$ on a plane, and that behaves in the usual way of electromagnetism and fulfill the condition

$$\oint \mathbf{a}(\mathbf{r}) \cdot d\mathbf{x} = \Phi. \tag{3.3}$$

To imagine an anyon consider a charged particle orbiting an infinitesimal solenoid. This charged particle must be on a plane to be considered a two dimensional system. By stating this, every anyon must have an specific magnetic vector potential such that the particle can not interact with the field. This magnetic field must be always non zero inside the solenoid but zero outside.

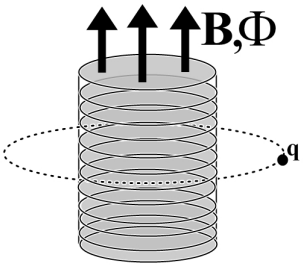


FIGURE 3.2: A charged particle orbiting a Solenoid

The proposed gauge for this system is named the anyon gauge and it is explicitly written in concordance with Equation 3.3 as

$$\mathbf{a}(\mathbf{r}) = \frac{\Phi}{2\pi} \frac{\hat{\mathbf{z}} \times \mathbf{r}}{|\mathbf{r}|^2}. \tag{3.4}$$

This gauge choice allows the magnetic field to be a function of $\mathbf{r}$ such that as $\mathbf{r} \to 0$ the magnetic field inside the solenoid is well defined. Since the solenoid is infinitesimal for practical and physical purposes, one has that the magnetic field must be $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{a}(\mathbf{r}) = \Phi\delta(\mathbf{r})\hat{\mathbf{z}}$. This also recovers the flux in Equation 3.3 by using the stokes theorem. The quantum condition of the magnetic field to be zero everywhere and non-zero inside the solenoid is correctly imposed by the chosen gauge, hence the anyonic gauge is a good choice in the description of this system. Recalling that this system is low dimensional we can use cylindrical coordinates for usefulness. To make a good description, suppose that the z component is so small that it is negligible. Then, Equation 3.4 can be written as

$$\mathbf{a}(\mathbf{r}) = \frac{\Phi}{2\pi}\frac{\hat{\mathbf{z}}\times\mathbf{r}}{|\mathbf{r}|^2} = \frac{\Phi}{2\pi}\frac{r}{r^2}\hat{\varphi} = \frac{\Phi}{2\pi r}\hat{\varphi}. \tag{3.5}$$

As one would expect, the azimuth component of the anyonic potential is the only non zero. Hence the movement of the charged particle is two dimensional, and it is in the plane perpendicular to the magnetic field. This gauge also accomplishes the classical and quantum impositions pointed out by the classical electromagnetism laws and the Aharonov-Bohm effect. As the main problem here is that the space is multiply-connected, a description of the system must be done under the considerations of the path integral formalism, as it was defined before. The focus is then the classical action integral. The lagrangian for this configuration is

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}}\cdot\mathbf{a}(\mathbf{r}), \tag{3.6}$$

this supposes that the particle has mass and non-zero kinetic energy. It is important to mention here that the particle is considered as a free particle. This due to the fact that as it was pointed out in Equation 2.1, a gauge transformation in the lagrangian requires an introduction of a quantity which is a function of the coordinates and the time. Also this function must be a derivative. Equation 3.6 it has a quantity that accomplishes this, that is, $q\dot{\mathbf{r}}\cdot\mathbf{a}(\mathbf{r})$. Since

$$q\dot{\mathbf{r}}\cdot\mathbf{a}(\mathbf{r}) = \frac{q\Phi}{2\pi}\frac{\dot{\mathbf{r}}\cdot\hat{\varphi}}{r} = \frac{q\Phi}{2\pi}\frac{r\dot{\varphi}}{r} = \frac{q\Phi}{2\pi}\dot{\varphi} = \frac{q\Phi}{2\pi}\frac{\mathrm{d}\varphi}{\mathrm{d}t}, \tag{3.7}$$

hence replacing this equation into Equation 3.6 gives

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{q\Phi}{2\pi}\frac{\mathrm{d}\varphi}{\mathrm{d}t}, \tag{3.8}$$

which is a gauge transformation for the lagrangian(Fasano & Marmi, 2006). It leaves the equation of motion unchanged. But more important, the movement is considered a free movement as

$$\mathbf{p} = \frac{\partial L}{\partial\dot{\mathbf{r}}} = m\dot{\mathbf{r}} + q\mathbf{a}(\mathbf{r}), \tag{3.9}$$

and the Hamiltionan function becomes

$$H = \mathbf{p}\cdot\dot{\mathbf{r}} - L = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{a}(\mathbf{r})\right)^2 = \frac{1}{2}m\dot{\mathbf{r}}^2, \tag{3.10}$$

which is the hamiltonian for a free particle. The action functional then has an overall term that is a gauge transformation. Thus this term can be taken out as a new action and considered as a phase. This procedure may look similar to that done in Chapter 2. And indeed it treats the exactly same system but by the difference of introducing a new gauge. In fact the action then takes a different form

$$S'[\varphi(t)] = S[\varphi(t)] + \phi[\varphi(t)]. \tag{3.11}$$

So, using the path integral formalism one arrives to

$$K(\varphi, t_b; 0, t_a) = \int \mathcal{D}[\varphi] \exp\left\{\frac{i}{\hbar}\int dt \left\{\frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{q\Phi}{2\pi}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right\}\right\}. \tag{3.12}$$

As the term inside the action integral is a gauge transformation for the lagrangian, it can be taken out of the amplitude without affecting the overall result. Resulting in

$$K(\varphi, t_b; 0, t_a) = \exp\left\{i\frac{q\Phi}{\hbar 2\pi}\int d\varphi\right\}\int \mathcal{D}[\varphi] \exp\left\{\frac{i}{\hbar}\int dt \frac{1}{2}m\dot{\mathbf{r}}^2\right\}. \tag{3.13}$$

The classical limits of integration ranges from an initial time to an end time. Let us say it is from $t_a \to t_b$, and the azimuth angle from $0 \to \varphi$. The limits of integration are chosen so that the particle can wind around the solenoid. As it is just one anyon, this phase means the winding of the particle around the solenoid behaving as a free particle. This property will be studied better for a symmetry exchange.

3.1 Statistics parameter in a symmetry exchange

As it was pointed out in the last section, an anyon could be portrayed as an enough long solenoid. That is infinitesimal and comprises a magnetic field only inside the solenoid. The Aharonov-Bohm effect is then necessary in order to make this description. But this thesis deals with the strange statistical parameter that arises under a symmetry exchange (exchange of two particles adiabatically), and how this parameter can be described. Then, as in Chapter 2 we need to treat a system of indistinguishable particles in the need for a multiply connected space. And then, consider a system composed with only just two particles $N = 2$. At first sight it seems a strange way to treat anyons. Since although a phase appeared it does not relate to the symmetry exchange. The way to implement this is to introduce the generalized gauge potential

for anyonic systems. This gauge arises from the last description and from the usefulness of the gauge transformation in the lagrangian. The general term for the gauge potential should be at least two dimensional and must contain the other particles potential acting on every particle. The proposed potential is

$$\mathbf{a}(\mathbf{r_i}) = \frac{\Phi}{2\pi} \sum_{j \neq i}^{N} \frac{\hat{\mathbf{z}} \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2}. \tag{3.14}$$

Now, remembering what was done before in chapter 2, the multiply connectedness of a system of indistinguishable particles lead to a different fundamental group for low dimensional systems. Anyons as proposed by Wilczek are particles that are orbiting an infinitesimal solenoid. So, for a many particle system the physical interpretation is that anyons are charged particles along with a magnetic flux.

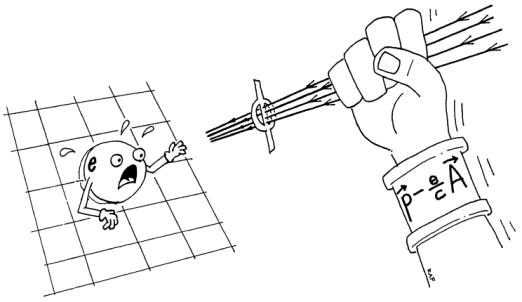


FIGURE 3.3: A charged particle in two dimensions, which possesses an associated flux induced by a magnetic potential (Wilczek & Shapere, 1989)

Also, anyons are indistinguishable particles. Then the configuration space is multiply connected (Chapter 2). Recalling what was done before, the introduction of this gauge potential must add a phase to the wave function as has already found, this gauge indeed accomplish the phase proposed by Wu (1984a). To see clearly this, let us treat the path integral for a two dimensional system composed with two particles. The lagrangian is

$$L = \frac{1}{2} m \dot{\mathbf{r}}_1^2 + \frac{1}{2} m \dot{\mathbf{r}}_2^2 + q \dot{\mathbf{r}}_1 \cdot \mathbf{a}(\mathbf{r_1}) + q \dot{\mathbf{r}}_2 \cdot \mathbf{a}(\mathbf{r_2}). \tag{3.15}$$

Recalling Chapter 2 for a two particles system, the configuration space is the Cartesian product of a center of mass space and a relative configuration space. The relative configuration space holds the properties of being indistinguishable and multiply connected as written in Equation 2.19. So, the lagrangian must be invariant under the action of the symmetric group. And it must not include the singular point. Thus, the lagrangian transforms as

$$L = \frac{1}{2}m\dot{\mathbf{r}}_1^2 + \frac{1}{2}m\dot{\mathbf{r}}_2^2 + \frac{q\Phi}{2\pi}\left[\dot{\mathbf{r}}_1 \cdot \left(\frac{\hat{\mathbf{z}} \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\right) + \dot{\mathbf{r}}_2 \cdot \left(\frac{\hat{\mathbf{z}} \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^2}\right)\right], \tag{3.16}$$

introducing the center of mass coordinate $\mathbf{R} \in \mathbb{R}$. And the relative movement coordinate $\mathbf{r} \in r(2,2)$. The result is

$$L = \frac{1}{2}m\dot{\mathbf{r}}_1^2 + \frac{1}{2}m\dot{\mathbf{r}}_2^2 + \frac{q\Phi}{2\pi}\left[2(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2) \cdot \left(\frac{\hat{\mathbf{z}} \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\right)\right], \tag{3.17}$$

$$L = m\dot{\mathbf{R}}^2 + \frac{1}{4}m\dot{\mathbf{r}}^2 + \frac{q\Phi}{\pi}\left[\dot{\mathbf{r}} \cdot \left(\frac{\hat{\mathbf{z}} \times \mathbf{r}}{|\mathbf{r}|^2}\right)\right]. \tag{3.18}$$

In order to account the rotation of one particle around the other particle, there must be a transformation of coordinates $(\mathbf{R}, \mathbf{r}) \to (R, \vartheta; r, \varphi)$, where ϑ is the azimuth movement of the center of mass of particles, and it is different from φ which gives account for the rotation of one particle around the other particle. This seems trivial, but it is important in the time for describing the phase. As before, using the path integral for this system the action changes in a phase. For which the amplitude becomes

$$K(\varphi, t_b; 0, t_a) = \int \mathcal{D}[\varphi] \exp\left\{\frac{i}{\hbar}\int dt \left\{m\dot{\mathbf{R}}^2 + \frac{1}{4}m\dot{\mathbf{r}}^2 + \frac{q\Phi}{\pi}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right\}\right\}. \tag{3.19}$$

Since the last term is only a phase that do not affect the equations of motion, the amplitude becomes

$$K(\varphi, t_b; 0, t_a) = \exp\left\{i\frac{q\Phi}{\hbar\pi}\int d\varphi\right\}\int \mathcal{D}[\varphi] \exp\left\{\frac{i}{\hbar}\int dt\ (m\dot{\mathbf{R}}^2 + \frac{1}{4}m\dot{\mathbf{r}}^2)\right\}. \tag{3.20}$$

Equation 3.20 is the overall result of this entire work. Since by comparison with Equation 2.35 this is the phase that arises when a symmetry exchange is done on a low dimensional system composed of two particles. In Equation 2.35, θ was a phase parameter which was unknown. But here, the phase parameter is found to be $\theta = \frac{q\Phi}{\hbar}$. For simplicity let us say that $\theta/\pi = \Upsilon$. Now, our problem is easily solved. Consider a wavefunction for two particles which are anyons, that means that this wavefunction is defined on the configuration space given by Equation 2.19

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) : \frac{\mathcal{Q}^2}{S_2} \longrightarrow \mathbb{C}. \tag{3.21}$$

Transforming the coordinates to the center of mass coordinates, and the relative motion coordinates the wavefunction transforms as $\Psi(\mathbf{r}_1, \mathbf{r}_2) \longrightarrow \Psi(R, \vartheta; r, \varphi)$. Since the center of mass coordinates are invariant under the action of S_2, from now on this coordinates will not be taken into account for calculations. To make a symmetry exchange one has to move both anyons from $0 \to \pi$ that is to move them from one point to its diametrically opposed point.

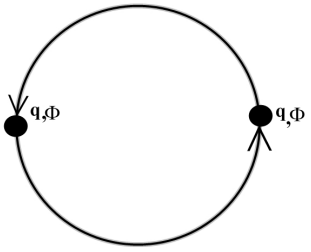


FIGURE 3.4: Anyons Exchanging their positions

As a result of this the wave function acquires a phase according to Equation 3.20 in the following way

$$\Psi(R, \vartheta; r, \varphi + \pi) = \exp\left\{ i \frac{q\Phi}{\hbar\pi} \int_0^{\pi} d\varphi \right\} \Psi(R, \vartheta; r, \varphi), \tag{3.22}$$

which when replacing Υ changes to

$$\Psi(R, \vartheta; r, \varphi + \pi) = \exp\left\{ i\Upsilon\pi \right\} \Psi(R, \vartheta; r, \varphi). \tag{3.23}$$

If somehow $\Upsilon = 0$, then particles will be bosons and will obey bosonic statistics. If it happens that $\Upsilon = 1$, then particles will be fermions and will obey fermionic statistics. But since $\Upsilon = q\Phi/\hbar\pi$ is a continuous function depending upon the charge and the flux, the statistics that the particle obeys is in between both statistics, that is named anyonic statistics. In general, if we have a system of N indistinguishable particles, one would have an specific phase for the wave function and the anyonic statistics. Although this problem seem very hard to solve, the proposed solution was found the same year that the solution for two particles, and it is as follows. According to Wu (1984b), as the main term in Equation 2.35 possesses a sum in the exponential, then one would have

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \exp\left\{ i \frac{\theta}{\pi} \int dt \frac{d}{dt} \left(\phi_{12}(t) + \phi_{23}(t) + \cdots + \phi_{(N-1)N}(t) \right) \right\} \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N). \tag{3.24}$$

With basic algebra one can write this phase as a product of individual phases, relating each symmetry exchange. One then obtains the general formula for the statistical phase of a system of N indistinguishable particles

$$\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\prod_{i<j}^{N}\exp\left\{i\frac{\theta}{\pi}\int dt(\frac{d}{dt}\phi_{ij}(t))\right\}\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N). \tag{3.25}$$

3.2 Quantum Mechanics of anyonic particles

Having studied the statistics of anyons, the result was the emergence of a non-trivial phase due to the considerations of low dimensional systems and indistinguishable particles. The overall phase was found under an exchange of particles so that the parameter was found. Now, it will be important to study the underlying physics of anyonic systems. That is, to study the quantum mechanics of low dimensional systems that presents anyonic statistics. To do this, only quantum mechanics of two particles will be studied since it is difficult to solve problems of many particles. Also, this study include the problem of two anyons under an harmonic potential due to its importance in physics, and then the study of the potential associated to a magnetic field that it is kept constant. But in order to do this let us study the quantum mechanics for "Free" anyons.

3.2.1 Quantum Mechanics of Two Free Anyons

In order to study free anyons, recall from the last section the gauge introduced for anyons. The two-particles lagrangian is like Equation 3.15. Introducing the center of mass coordinates and the relative motion coordinate, the lagrangian transforms as Equation 3.18. Also, this problem simplifies significantly by using cylindrical coordinates with the z component zero. One can write the lagrangian as $L = L_R + L_r$, where L_R gives account for the center of mass motion. As this motion is invariant under the action of the symmetric group S_2 this part of the lagrangian is unimportant in this description. Then, the focus here is on L_r. The lagrangian can be written as

$$L_r=\frac{1}{4}m(\dot{r}^2+r^2\dot{\varphi}^2)+\hbar\Upsilon\dot{\varphi}. \tag{3.26}$$

In order to quantize this system one has to find the hamiltonian of the system, which is given by the legendre transformation

$$H = \mathbf{p}_r \cdot \dot{\mathbf{r}} + \mathbf{p}_\varphi \cdot \dot{\boldsymbol{\varphi}} - L. \tag{3.27}$$

Using the hamilton equations one obtains that

$$\dot{\boldsymbol{\varphi}} = \frac{2}{mr^2}(p_\varphi - \hbar\Upsilon)\hat{\varphi}, \tag{3.28}$$

$$\dot{\mathbf{r}} = \frac{2}{m}p_r\hat{r}, \tag{3.29}$$

after replacing and operating in Equation 3.27 one obtains that the hamiltonian is

$$H = \frac{p_r^2}{m} + \frac{(p_\varphi - \hbar\Upsilon)^2}{mr^2}. \tag{3.30}$$

To quantize one, replaces the momentum by the quantum mechanical operator, and introduces that into the Schrödinger equation. One remark to add is that, since the wavefunction is an overall function of the relative movement space and from the center of mass space, the wavefunction is written as $\Psi(R, \vartheta; r, \varphi)$. Since the lagrangian is separable, the hamiltonian is separable, then the wavefunction is also separable. By this argument the wavefunction it is written as

$$\Psi(R, \vartheta; r, \varphi) = \psi_R(R, \vartheta)\psi_r(r, \varphi), \tag{3.31}$$

as mentioned before. The overall wavefunction must contain the solution of the Schrödinger equation for the center of mass coordinate, but since it is invariant under the action of the symmetric group, then one has that this coordinate does not contribute to the description since it will be the well known solution for a system with angular momentum(Le Bellac, 2011, p. 318). In this description only the relative motion wavefunction will be completely solved. According to what was said before, the Schrdöinger equation is

$$\left(\frac{\hat{p_r}^2}{m} + \frac{(\hat{p_\varphi} - \hbar\Upsilon)^2}{mr^2}\right)\psi_r(r, \varphi) = E_r\psi_r(r, \varphi), \tag{3.32}$$

replacing the momentum operators the Schrödinger equation becomes

$$\left(-\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{\hbar^2}{mr^2}\left(i\frac{\partial}{\partial\varphi}+\Upsilon\right)^2\right)\psi_r(r,\varphi)=E_r\psi_r(r,\varphi). \tag{3.33}$$

As usual, supposing that the solution of $\psi_r(r,\varphi)$ is separable, one can state this as $\psi_r(r,\varphi)=\tilde{r}(r)\tilde{\varphi}(\varphi)$, therefore one obtains

$$\left(i\frac{\partial}{\partial\varphi}+\Upsilon\right)^2\tilde{r}(r)\tilde{\varphi}(\varphi)=\frac{E_r m r^2}{\hbar^2}\tilde{r}(r)\tilde{\varphi}(\varphi)+r^2\frac{\partial^2(\tilde{r}(r)\tilde{\varphi}(\varphi))}{\partial r^2}+r\frac{\partial(\tilde{r}(r)\tilde{\varphi}(\varphi))}{\partial r}, \tag{3.34}$$

$$-\frac{\partial^2(\tilde{r}\tilde{\varphi})}{\partial\varphi^2}+i\Upsilon\frac{\partial(\tilde{r}\tilde{\varphi})}{\partial\varphi}+\Upsilon^2\tilde{r}\tilde{\varphi}=\frac{E_r m r^2}{\hbar^2}\tilde{r}\tilde{\varphi}+r^2\frac{\partial^2(\tilde{r}\tilde{\varphi})}{\partial r^2}+r\frac{\partial(\tilde{r}\tilde{\varphi})}{\partial r}. \tag{3.35}$$

Dividing on both sides of Equation 3.35 by $\tilde{r}(r)\tilde{\varphi}(\varphi)$

$$\frac{1}{\tilde{\varphi}}\left(-\frac{\partial^2(\tilde{\varphi})}{\partial\varphi^2}+i\Upsilon\frac{\partial(\tilde{\varphi})}{\partial\varphi}+\Upsilon^2\tilde{\varphi}\right)=\frac{E_r m r^2}{\hbar^2}+r^2\frac{\partial^2(\tilde{r})}{\partial r^2}+r\frac{\partial(\tilde{r})}{\partial r}. \tag{3.36}$$

One then obtains that, as every side of Equation 3.36 is independent of the other coordinate, the equation of motion must be a constant. So, taking the left hand side equation

$$-\frac{\partial^2(\tilde{r}\tilde{\varphi})}{\partial\varphi^2}+i\Upsilon\frac{\partial(\tilde{r}\tilde{\varphi})}{\partial\varphi}+\Upsilon^2\tilde{r}\tilde{\varphi}=\eta\tilde{\varphi}. \tag{3.37}$$

In order to solve this equation, note that as the phase during an exchange is obtained by adding π to the relative motion azimuth angle, the overall wave function remains the same as in Equation 3.31. Then $\tilde{\varphi}(\varphi+\pi)=\tilde{\varphi}(\varphi)$. The only solution to Equation 3.37 that fulfills this boundary condition is

$$\tilde{\varphi}(\varphi)=\exp\{il\varphi\}, \tag{3.38}$$

where $l=2n$ with $n\in\mathbb{Z}$. By introducing Equation 3.38 in Equation 3.37 the eigenvalues obtained are $\eta=(l-\Upsilon)^2$. So, the general wavefunction solution comes from rearranging Equation 3.36 and introducing the resultant eigenvalues

$$0=\left[\frac{E_r m r^2}{\hbar^2}-(l-\Upsilon)^2\right]\tilde{r}+r^2\frac{\partial^2(\tilde{r})}{\partial r^2}+r\frac{\partial(\tilde{r})}{\partial r}. \tag{3.39}$$

By inspection of Equation 3.39 one can see that this is the bessel differential equation. Thus the solutions to this differential equation are the bessel functions of first kind(Arfken, Weber, & Harris, 2011, p. 645). Hence, replacing $\xi^2 = mE_r/\hbar^2$ for usefulness, the solutions for $\tilde{r}(r)$ are

$$\tilde{r}(r) = J_{|l-\Upsilon|}(r\xi). \tag{3.40}$$

Then, the wavefunction for this system is found by replacing in Equation 3.31

$$\Psi(R, \vartheta; r, \varphi) = \psi_R(R, \vartheta) e^{il\varphi} J_{|l-\Upsilon|}(r\xi). \tag{3.41}$$

One can see how the wavefunction for an anyonic system is indeed the wavefunction for a system of fermions or bosons, upon choosing $\Upsilon = 1$ or $\Upsilon = 0$. Thus, the anyonic description for two free anyons is indeed in between both descriptions.

3.2.2 Quantum Mechanics of two anyons under an harmonic potential

If a harmonic potential is considered in this system, the overall lagrangian which describes it changes only by the addition of the harmonic term $V(r) = m\omega^2 r^2/2$ for each particle. Then, after transforming into the relative motion and center of mass motion the overall lagrangian is

$$L = m\dot{\mathbf{R}}^2 + \frac{1}{4}m\dot{\mathbf{r}}^2 + \hbar\Upsilon\dot{\varphi} - \frac{1}{4}m\omega^2 r^2 - m\omega^2 R^2. \tag{3.42}$$

This Lagrangian as before can be separated into center of mass lagrangian and relative motion lagrangian. The center of mass description is the same as for any quantum harmonic oscillator but with the difference of having double of the mass. The relative motion description is given by the lagrangian

$$L_r = \frac{1}{4}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \hbar\Upsilon\dot{\varphi} - \frac{1}{4}m\omega^2 r^2. \tag{3.43}$$

As the harmonic potential is independent of the speed, the hamiltonian is the same as in Equation 3.30 but with the addition of the potential energy

$$H = \frac{p_r^2}{m} + \frac{(p_\varphi - \hbar\Upsilon)^2}{mr^2} + \frac{1}{4}m\omega^2 r^2. \tag{3.44}$$

The wavefunction is separable into the azimuth and radial components. As before, the Schrödinger equation just have one additional term (the harmonic potential)

$$\left(i\frac{\partial}{\partial\varphi}+\Upsilon\right)^2\tilde{r}\tilde{\varphi}=\frac{E_r m r^2}{\hbar^2}\tilde{r}\tilde{\varphi}+r^2\frac{\partial^2(\tilde{r}\tilde{\varphi})}{\partial r^2}+r\frac{\partial(\tilde{r}\tilde{\varphi})}{\partial r}+\tilde{r}\tilde{\varphi}\frac{m^2\omega^2 r^4}{4\hbar^2}. \tag{3.45}$$

Since both sides are independent of each other, the same argument used before for free anyons applies for the azimuth angle in this case, in which the azimuth solution is Equation 3.38. The same goes for the eigenvalues, so the general differential equation is

$$0=\left[\frac{E_r m r^2}{\hbar^2}-(l-\Upsilon)^2-\frac{m^2\omega^2 r^4}{4\hbar^2}\right]\tilde{r}+r^2\frac{\partial^2(\tilde{r})}{\partial r^2}+r\frac{\partial(\tilde{r})}{\partial r}. \tag{3.46}$$

Indeed this is a hard differential equation to solve. But the answer has been found using a series of tricks. Following Khare (Khare, 2005, p. 48) the way to correctly solve this equation is to propose an ansatz that could leave the equation in the general form for one special function, for instance, Bessel, Laguerre, or Legendre functions. This ansatz is

$$\tilde{r}(r)=r^{|l-\Upsilon|}\exp\left\{-\frac{m\omega r^2}{4\hbar}\right\}f(r). \tag{3.47}$$

After introducing the proposed ansatz, the differential equation can be converted to

$$\frac{\partial^2 f}{\partial r^2}+\left(\frac{(2|l-\Upsilon|+1)}{r}-\frac{m\omega}{\hbar}r\right)\frac{\partial f}{\partial r}+\left(\frac{Em}{\hbar^2}-(|l-\Upsilon|+1)\frac{m\omega}{\hbar}\right)f=0. \tag{3.48}$$

To simplify Equation 3.48 let us use $g=m\omega r^2/2\hbar$. So,

$$g\frac{\partial^2 f}{\partial r^2}+(|l-\Upsilon|+1)-g)\frac{\partial f}{\partial r}+\frac{1}{2}\left(\frac{E}{\hbar\omega}-(|l-\Upsilon|+1)\right)f=0. \tag{3.49}$$

Equation 3.49 is the generalized Laguerre differential equation(Arfken et al., 2011, p. 889). The solutions then are the Laguerre polynomials

$$\tilde{r}(r)=r^{|l-\Upsilon|}\exp\left\{-\frac{m\omega r^2}{4\hbar}\right\}L_{n_r}^{|l-\Upsilon|}\left(\frac{m\omega}{2\hbar}r^2\right). \tag{3.50}$$

Having found the solutions, the final step is to write the complete wavefunction for this system. That is,

$$\Psi(R,\vartheta;r,\varphi)=\psi_R(R,\vartheta)e^{il\varphi}r^{|l-\Upsilon|}\exp\left\{-\frac{m\omega r^2}{4\hbar}\right\}L_{n_r}^{|l-\Upsilon|}\left(\frac{m\omega}{2\hbar}r^2\right). \tag{3.51}$$

Again a clear dependence on Υ is seen, differing clearly from the harmonic oscillator in higher dimensions.

3.2.3 Quantum mechanics of two anyons under an uniform magnetic field

As a very important example is the problem of two anyons on a magnetic field that is kept constant. For simplicity let us assume that the magnetic field has components only in the z-axis. But it is different from the magnetic field inside the solenoid of each particle. This is an external magnetic field given by $\mathbf{B}=B_0\hat{\mathbf{z}}$. This problem was solved by Johnson and Canright (1990); Karlhede and Westerberg (1992); Verçin (1991), the solutions proposed by them range from a hard solution to a really hard solution. So, the proposed solution here is found following the aforementioned procedures by starting form the lagrangian, and the following is to solve the Sc the full solution. It is important to state that the chosen gauge for the magnetic potential(external field) is the symmetric gauge according to the aforementioned authors. The symmetric gauge is

$$\mathbf{A}_i=\frac{1}{2}\mathbf{B}\times\mathbf{r}_i, \tag{3.52}$$

introducing this term into the lagrangian, and transforming to the center of mass and the relative motion. It becomes

$$L=m\dot{\mathbf{R}}^2+\frac{1}{4}m\dot{\mathbf{r}}^2+\hbar\Upsilon\dot{\varphi}+qB_0R^2\dot{\vartheta}+\frac{1}{4}qB_0r^2\dot{\varphi}. \tag{3.53}$$

As before this lagrangian can be separated into the relative and the center of mass motion, but as one can see better this problem it is already a hard problem, since the center of mass lagrangian also have a azimuth coordinate. The solutions to the center of mass coordinate are the well known Landau levels under the symmetric gauge. As this issue is important in the description of the fractional quantum hall effect, the solution will be postponed to the next chapter. Thus focusing on the relative motion coordinate. The hamiltonian is

$$H_r=\frac{p_r^2}{m}+\frac{\left(p_\varphi-\hbar\Upsilon-\frac{qB_0r^2}{4}\right)^2}{mr^2}. \tag{3.54}$$

Again considering the Schrodinger equation for this system, replacing the momentum by their quantum respective operators, and separating the wave equation one obtains the differential equation for the relative motion

$$\left(i\frac{\partial}{\partial\varphi}+\Upsilon+\frac{qB_0r^2}{\hbar 4}\right)^2\tilde{r}\tilde{\varphi}=\frac{E_rmr^2}{\hbar^2}\tilde{r}\tilde{\varphi}+r^2\frac{\partial^2(\tilde{r}\tilde{\varphi})}{\partial r^2}+r\frac{\partial(\tilde{r}\tilde{\varphi})}{\partial r}+\tilde{r}\tilde{\varphi}\frac{m^2\omega^2r^4}{4\hbar^2} \tag{3.55}$$

As was done before by separating this differential equation one founds that the azimuth coordinate is given again by Equation 3.38. So, using that solution for the azimuth angle one can convert the differential equation into

$$0=\left[\frac{E_rmr^2}{\hbar^2}+\frac{qB_0r^2}{2\hbar}(l-\Upsilon)-(l-\Upsilon)^2-\frac{q^2B_0^2r^4}{16\hbar^2}\right]\tilde{r}+r^2\frac{\partial^2(\tilde{r})}{\partial r^2}+r\frac{\partial(\tilde{r})}{\partial r} \tag{3.56}$$

The solution to this differential equation has a procedure similar to what was done before. The aforementioned authors used different methods to arrive to this equation. One of this methods is by the use of creation and annihilation operators. This solution will not be treated here since that is considered as the Landau problem which will be discussed in the description of the fractional quantum Hall effect. The general solution proposes that two wavefunctions are solution of this equation[1]

$$\Psi^I(R,\vartheta;r,\varphi)=\psi_R(R,\vartheta)e^{il\varphi}r^{|l-\Upsilon|}\exp\left\{-\frac{m\omega_cr^2}{4\hbar}\right\}L_{n_r}^{|l-\Upsilon|}\left(\frac{m\omega_c}{2\hbar}r^2\right) \tag{3.57}$$

for angular momentum values given by $l>0$, and

$$\Psi^{II}(R,\vartheta;r,\varphi)=\psi_R(R,\vartheta)e^{-i|l|\varphi}r^{|l|+\Upsilon}\exp\left\{-\frac{m\omega_cr^2}{4\hbar}\right\}L_{n_r}^{|l|+\Upsilon}\left(\frac{m\omega_c}{2\hbar}r^2\right), \tag{3.58}$$

if $l\leq 0$. Although this solutions seems strange on first sight. It is clear that this must be the solutions of this problem, since up to now no energy values had been found, the energy values clearly show differences between normal particles(bosons or fermions) and anyons. That is what is going to be found.

[1] Upon depending on the energy values

3.2.4 Energy values in the anyonic quantum mechanics

In order to give a concrete description of the anyonic quantum mechanics, one has to find the eigenvalues associated to the hamiltonian. As a similar procedure to the well known (Diagonalizing the hamiltonian) is made, for simplicity only the solution is shown.

1. **Energy values for two free anyons.**

   Since the bessel functions are orthonormal under integration, to find the mean value of the energy one finds that as the anyons are free particles the energy must be that for a free particle. Then, by one finds that the energy for two free anyons must be

   $$E = \frac{\hbar^2 \xi^2}{m}, \tag{3.59}$$

   which is indeed the energy of a free particle. So, in the anyon gauge the particles are free, but the wavefunction is still in between bosons and fermions.

2. **Energy values for two anyons under a harmonic potential**

   In this case since the particles are treated as free anyons under an harmonic potential, one would expect to find that the energy values are that for a quantum harmonic oscillator. By integrating, one finds according to (Wilczek, 1990, p. 20) that

   $$E_{n_r,l} = \hbar\omega(2n_r + |l - \Upsilon| + 1). \tag{3.60}$$

   If it happens that $l = 0$ one would recover the energy spectrum for bosons under $\Upsilon = 0$, and fermion spectrum under $\Upsilon = 1$. One can see that this spectrum is degenerate since in general Υ is a continuous function.

3. **Energy values for two anyons under an uniform magnetic field.**

   As done before integrating and obtaining the mean value for the hamiltonian, one finds the energy for this case (Verçin, 1991). It happens that by taking the general solution the energy values are

   $$E_{n_r,l} = \hbar\omega(2n_r + |l - \Upsilon| - (l - \Upsilon) + 1). \tag{3.61}$$

   Here, if $l > 0$ the energy values are

   $$E^I_{n_r,l} = \hbar\omega(2n_r + 1). \tag{3.62}$$

If $l \leq 0$, then the energy values are

$$E^{II}_{n_r,l} = \hbar\omega(2n_r + 2|l| + 1). \tag{3.63}$$

That is the main reason for two wavefunctions to be solutions to the Schrödinger equation. By simple inspection, every level in the first case is degenerate; maybe infinitely degenerate. For the other case, the energy levels are degenerate but according to Khare (2005) they are infinite degenerate for n_r being odd.

In each case the energy levels are computed searching for differences from bosons or fermions, the main difference comes from the anyonic phase factor of being continuous in between both statistics. So, considering the anyonic gauge indeed shows a very different quantum mechanics.

3.3 Spin of one anyon

In the previous sections the quantum mechanics for anyon particles was developed. Indeed, one can understand basically what happens when one consider flux tube charged particles. But a very important question may arise under this introduction: if anyons are particles that induce a phase under an exchange, and this phase might be in between both types of possible statistics, then, which is the spin for an anyon? Is ut half-integer or integer? To solve this questions, mathematically one can see that in three dimensions, the group that relates rotations is $SO(3)$. This group is non abelian. That means that the generators of this group do not commute. As the generators of this group do not commute one can have a lie algebra associated to this group namely $so(3)$. This algebra has non zero commutator and it is the underlying concept of the angular momentum operators. This means that in three dimensions the angular momentum has canonical commutation relationship (CCR) which leads to well known quantum mechanics. But in low dimensional systems, the rotation group is $SO(2)$ which is an abelian group. Thus the lie algebra associated is $so(2)$ which has zero commutator. Then the angular momentum does not has CCR. So, in low dimensional systems the spin is less restricted, a practically and useful way to see this is to consider one anyon. The lagrangian is as in Equation 3.8 and it satisfies the low dimensional requirement. Then, the canonical momentum is

$$\mathbf{p}_c = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + q\mathbf{a}(\mathbf{r}). \tag{3.64}$$

Considering the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p_c}$ one obtains

$$\mathbf{L} = \mathbf{r} \times (m\dot{\mathbf{r}}) + q\mathbf{r} \times \mathbf{a}(\mathbf{r}). \tag{3.65}$$

This angular momentum is on the z component by definition of the potential, then in components one obtains for the z component of angular momentum

$$L = L_z - \frac{q\Phi}{2\pi}, \tag{3.66}$$

as we know from quantum mechanics, the eigenvalues of the angular momentum operator in z are

$$L = \hbar l_z - \hbar 2\Upsilon, \tag{3.67}$$

as the total angular momentum is conserved by definition. Wilczek (1982a) defined the anyon spin as $s = -\hbar 2\Upsilon$. Spin in low dimensional systems then is not integer or half-integer, it is instead fractional and can have any value in between both statistics $\hbar(l_z - 2\Upsilon)$. Of course the overall description of spin for anyons is much more complex and relates quantum field theory and the Chern-Simons gauge. Here the discussion is qualitative and does not have such profound argument since the main objective of the thesis is to describe the main characteristics of low dimensional systems that possesses anyonic statistics. To finalize one may think: ok, anyons are a mathematical feature that arises in low dimensional systems, and are product of multiply connected spaces. But as physics needs to be proved under experimental conditions to acquire a truth value, then is this anyonic formalism just a mathematical sophistication of low dimensional systems, or does it really exists an application of the anyonic formalism? This is what is going to be considered in the next chapter.

Chapter 4

The Fractional Quantum Hall Effect and the Anyonic Term

The development made up to now is purely mathematical in some way, and theoretical in other way. The study of identical particles and its configuration space lead us to the property of being multiply connected, which in turn is a prerequisite for the fundamental group of this space to be the braid group in low dimensional systems. Also the multiply connectedness arises in the Aharonov-Bohm effect. A solenoid induces a phase into the wavefunction of a charged particle. Besides, no particle can travel inside the solenoid, and therefore, it causes the space to be multiply connected. Upon this idea, the dynamical realization of anyon systems is constructed, and it reaches a point where the mathematical description and the dynamical description converge. Even though all this theoretical development seems true, physics is about facts. Therefore, we need an empirical result to prove that this new physics is true. This empirical result comes from the fractional quantum hall effect which will be studied in this chapter with the anyonic formalism to prove the application of this theoretical development.

4.1 The Hall effect

4.1.1 Classical Hall effect

The classical hall effect made simple is the production of a voltage on a thin plate (a semiconductor) named hall voltage. This voltage is transverse to an external magnetic field, and also is transverse to an electric field which produces an electric current(Patterson & Bailey, 2007,

p. 311). The classical explanation is very simple. Consider a very thin plate and an electrical current for simplicity in the x-axis. When one induce a magnetic field on the z-axis, the electrons that are moving on the x-axis curve their trajectory according to Lorentz law force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. Because of this curving on the electron's path, on one side of the plate the excess of electrons generate an electric field opposite to the Lorentz force. Then, the forces reach the equilibrium if $eE_y = ev_xB_z$. Hall defined a coefficient such that one can measure the relation between the electric field and the magnetic field in the following way

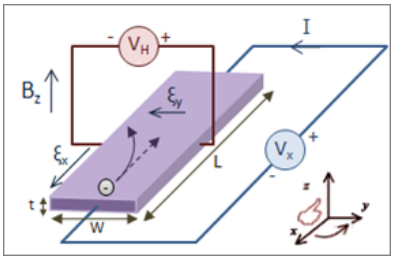


FIGURE 4.1: Schematic view of the hall effect(Wikipedia, 2016)

$$R_H = \frac{E_y}{i_x B_z} = -\frac{1}{n_e e}. \tag{4.1}$$

Here n_e is the charge carriers density. With this result, one can measure the charge carrier density, the magnetic field, and the hall voltage which results to be $V_H = -E_y w$. One very important feature of the Hall effect is that it differentiates between positive charges moving in one direction and negative charges moving in the opposite. The Hall effect offered the first real proof that electric currents in metals are carried by moving electrons, not by protons. The Hall effect also showed that in some substances, it is more appropriate to think of the current as positive "holes" moving rather than negative electrons. Another important thing to mention is that the overall current density is given classically by $i = -n_e ev$. The argument to show the current density is that this result will be important in the quantum version of the Hall effect.

4.1.2 The quantum Hall Effect

In studying semiconducting hetero-structures subject to high magnetic fields and low temperatures (about $2K$), the quantized version of the hall effect was discovered (F. D. M. Haldane, 1983; Klitzing, Dorda, & Pepper, 1980; Tsui, Stormer, & Gossard, 1982; Von Klitzing, 1986). The experiment leading to this well known effect consisted of a thin layer in between two semiconductors. Under a high magnetic field and low temperature the thin layer electrons freeze on

the direction parallel to the magnetic field. Thus the movement of the electrons can be though as two dimensional. Indeed the idealization is for a two dimensional gas of electrons.

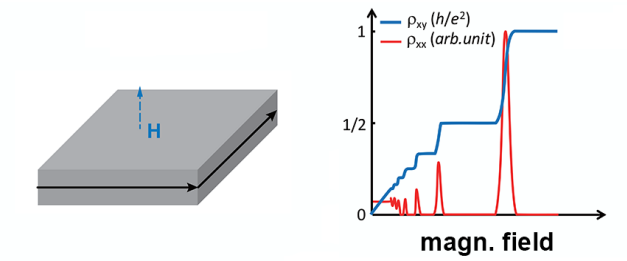


FIGURE 4.2: The quantum hall effect

As a result the Hall conductivity and resistivity quantize in accordance with the Landau levels problem (Goerbig, 2009). In fact, it happens that the Hall conductivity results to be a multiple of a quantity. So, the resistivity is also a multiple of a quantity (one is the inverse of the other). It happened that the Hall conductivity was $\sigma_H = \nu e^2/h$. It is worthwhile to mention that the Hall conductivity is different from the usual conductivity, since the conductivity is in essence a second order tensor that is anti-symmetric. In general the electric current density is given by $i_m = \sigma_{nm} E_n$. Here σ_{nm} is the conductivity tensor, the diagonal of this tensor σ_{nn} is named the longitudinal conductance, while the other components are the Hall conductance σ_{nm}. For two dimensional systems the longitudinal conductance is $\sigma_{xx} = \sigma_{yy} = 0$, while the hall conductance is $\sigma_{xy} = -\sigma_{yx}$. The Hall resistivity is thus the inverse tensor of the conductance $\rho_{nm} i_m = E_n$, then $\rho_H = h/\nu e^2$. As in Figure 4.2 the hall resistivity has some plateaus on specific values of magnetic field. This is due to the prefactor ν named the filling factor.

One may wonder about the specific value of the filling factor ν. The fact is that the filling factor determines the values of the resistivity, and experimentally it was found to be independent of the material itself. Theoretically the filling factor is defined as the number of electrons per number of Landau levels available, the precise definition of Landau levels will be given later. It was discovered that this filling factor could be like spin, it could have integer values of specific constants or fractional values. If ν is integer, the name for this effect is integer quantum Hall effect. If ν is fractional, the name for this effect is fractional quantum hall effect. Experimentally for the case of integer values, the Hall resistivity was found to have plateaus on specific values of magnetic field, but the longitudinal resistivity is zero in that values as in Figure 4.3. For the case of fractional values of the filling factor the Hall resistivity is split in fractional values, also depending upon the specific values of the magnetic field like in Figure 4.4.

At the time of discovery, the quantum hall effect was a really hard problem to understand. The description was given by F. D. M. Haldane (1983); R. Laughlin (1984); R. B. Laughlin

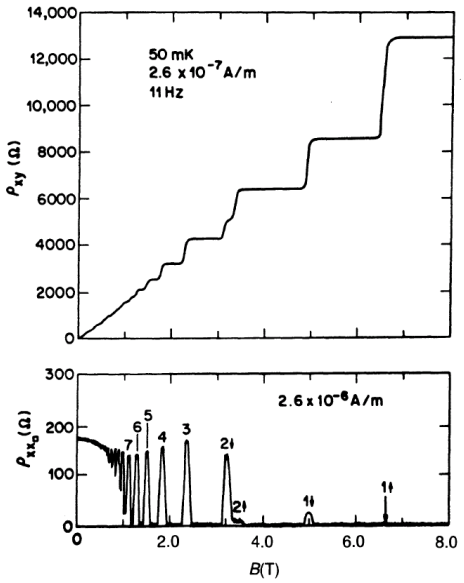


FIGURE 4.3: Hall resistivity for integer values of the filling factor(Yoshioka, 2013, P. 11)

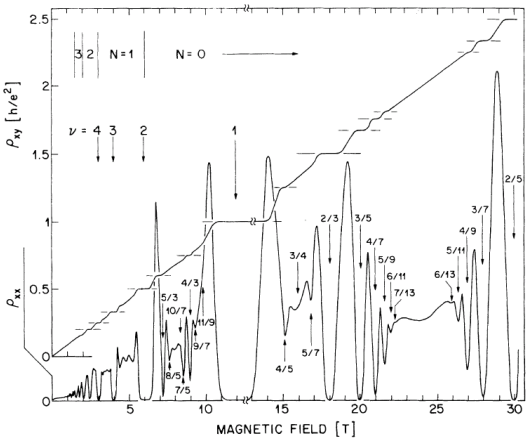


FIGURE 4.4: Hall resistivity for fractional values of the filling factor(Yoshioka, 2013, P. 15)

(1983). The main problem seems to be the quantization of a low dimensional system subject to a magnetic field. This is the Landau levels problem and it was solved long ago.

4.1.3 Landau Levels Problem

In order to understand the quantum Hall effect one must deal with the Landau levels problem. The Landau problem consist on determining how does a system of a single particle on the plane behaves in presence of a magnetic field which is held constant.

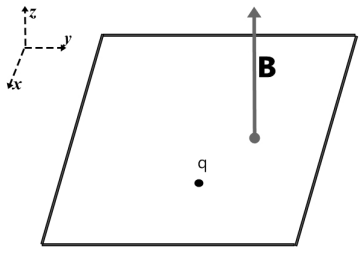


FIGURE 4.5: Free particle on a plane subject to a constant magnetic field

As we know, in a plane where there is a charged particle and a magnetic field perpendicular to it, a force acts on the charged particle that is the Lorentz force. In cartesian coordinates $(q_1, q_2) = (x, y)$ we can obtain two differential equations for the charged particle, their solutions describe the classical movement of the electron

$$\begin{aligned} \ddot{x} &= -\tfrac{e}{m_e}\dot{y}|B|, \\ \ddot{y} &= \tfrac{e}{m_e}\dot{x}|B|. \end{aligned} \tag{4.2}$$

Taking $e|B|/m_e = \omega_c$ usually called the cyclotron frequency, and solving the system of differential equations we get

$$\begin{aligned} (x(t) - x_0) &= r\cos(\omega_c t), \\ (y(t) - y_0) &= -r\sin(\omega_c t). \end{aligned} \tag{4.3}$$

Equation 4.3 describe the movement of the electron in the plane, if there exists electric field on this system the problem becomes difficult because it will depend on the explicit form of the electric field, and the equations to solve will be

$$\begin{aligned} \ddot{x} &= -\omega_c\dot{y} + \tfrac{e}{m_e}E_x, \\ \ddot{y} &= \omega_c\dot{x} + \tfrac{e}{m_e}E_y. \end{aligned} \tag{4.4}$$

The quantum formulation of this problem requires to define a momentum operator to simplify the calculations(Goerbig, 2009). In this case we will choose the canonical momentum. So, we define the following momentum operators which depend on the chosen gauge and will be named kinetic momentum

$$\begin{aligned}\hat{\pi}_x &= \hat{p}_x - e\hat{A}_x, \\ \hat{\pi}_y &= \hat{p}_y - e\hat{A}_y.\end{aligned} \tag{4.5}$$

This operators do not commute that is $[\hat{\pi}_x, \hat{\pi}_y] \neq 0$ and this commutator do not depend on the chosen gauge. It depends on $|\mathbf{B}|$. It results that $[\hat{\pi}_x, \hat{\pi}_y] = i\hbar e B_z$. The other feature of using this momentum operators, is that the coordinates of the center of the cyclotron movement can not be known with precision, since they do not commute $[x_0, y_0] \neq 0$. Their commuter is equal to $[\hat{x}_0, \hat{y}_0] = -i\hbar/\omega_c m_e$. If we assume that the operators $\hat{\pi}_x, \hat{\pi}_y$ play the role of the position and momentum operator respectively, it is possible to think that the system can be treated as a harmonic oscillator, since the formulation will be the same; by defining a pair of creation and annihilation operators $\hat{a}, \hat{a}^\dagger$ in the following way

$$\hat{a} = \frac{1}{\sqrt{2e\hbar|\mathbf{B}|}}(\hat{\pi}_x + i\hat{\pi}_y) = \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x + i\hat{\pi}_y), \tag{4.6}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2e\hbar|\mathbf{B}|}}(\hat{\pi}_x - i\hat{\pi}_y) = \frac{l}{\sqrt{2}\hbar}(\hat{\pi}_x - i\hat{\pi}_y), \tag{4.7}$$

with $l = \sqrt{\hbar/eB}$ called the larmor radius. It is easy to solve for $\hat{\pi}_x, \hat{\pi}_y$, so that the Hamiltonian of the system can be expressed in terms of the $\hat{a}^\dagger, \hat{a}$

$$\hat{\pi}_x = \frac{\hbar}{\sqrt{2}l}(\hat{a}^\dagger + \hat{a}), \tag{4.8}$$

$$\hat{\pi}_x = i\frac{\hbar}{\sqrt{2}l}(\hat{a}^\dagger - \hat{a}). \tag{4.9}$$

This leads to the well know hamiltonian for the harmonic oscillator

$$\hat{H} = \hbar\omega_c(\hat{a}^\dagger\hat{a} + 1/2). \tag{4.10}$$

The spectrum of this hamiltonian is that of a harmonic oscillator $E = \hbar\omega_c(N + 1/2)$. The eigenvalues of the hamiltonian will be named Landau levels. Now, to find the eigenfunctions we only have to solve $\hat{a}\,|0\rangle = 0$ in order to obtain the ground state wave-function of the system, but there is a slightly difference with the usual groundstate of a harmonic oscillator. That is, the system depends entirely on the choice of the gauge in $\mathbf{A}$. There exists basically two different

gauge choices for this problem, the first one is the Landau gauge, and the second one is the symmetric gauge. The first one preserves the rotational invariance, the second one preserves the translational invariance on the x-axis, or the translational invariance on the y-axis.

- **Symmetric gauge**

$$\mathbf{A} = (A_x, A_y) = \frac{B}{2}(-y, x)$$

- **Landau gauge**

$$\mathbf{A} = (A_x, A_y) = B(0, x)$$

$$\mathbf{A} = (A_x, A_y) = B(-y, 0)$$

Rotationally invariant gauge(symmetric gauge): Using the symmetric gauge (the one which is Rotationally invariant) $\mathbf{A} = B/2(-y, x)$ in the definition of the operator $\hat{a}$ one has

$$\hat{a} = \frac{l}{\sqrt{2}\hbar}\left(p_x + \frac{eB}{2}y + ip_y - i\frac{eB}{2}x\right), \tag{4.11}$$

$$\hat{a} = -\frac{il}{\sqrt{2}}\left(2\frac{\partial}{\partial z^*} + \frac{eB}{2\hbar}z\right). \tag{4.12}$$

Similarly, the creation operator in the symmetric gauge is

$$\hat{a}^\dagger = -\frac{il}{\sqrt{2}}\left(2\frac{\partial}{\partial z} - \frac{eB}{2\hbar}z^*\right). \tag{4.13}$$

The last relationships are recovered with $z = x + iy$ and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{4.14}$$

As usual one applies $\hat{a}\,|0\rangle = 0$ and put this into the basis $\langle z, z^*|$, and solve to get the ground state wave-function

$$\langle z, z^*|\,\hat{a}\,|0\rangle = \langle z, z^*| - \frac{il}{\sqrt{2}}\left(2\frac{\partial}{\partial z^*} + \frac{eB}{2\hbar}z\right)|0\rangle = 0, \tag{4.15}$$

$$\frac{il}{\sqrt{2}}\left(2\frac{\partial}{\partial z^*}\psi_o(z, z^*) + \frac{eB}{2\hbar}z\psi_o(z, z^*)\right) = 0, \tag{4.16}$$

$$\psi_o(z, z^*) = f(z)e^{\frac{-|z|^2}{4l^2}}. \tag{4.17}$$

There are some things to say about Equation 4.17. One of this things is that $f(z)$ is an arbitrary function. This suggests that there exist many ground states as analytic functions are. It is customary to choose $f(z)$ as function which is a polynomial in z. Namely $f(z) = N_n z^n$ for the general case, with N_n to be determined. Due the fact that this function is square integrable and can be normalized, one can retrieve from the normalization condition N_n in the following way

$$N_n^2 \int z^{2n} e^{\frac{-|z|^2}{4l^2}} d^2z = N_n^2 \int 2\pi e^{\frac{-r^2}{2l^2}} r^{2n+1} dr = 1, \tag{4.18}$$

$$N_n^2 \int \pi e^{\frac{-t}{2l^2}} t^n dt = N_n^2 \pi (2l^2)^{n+1} \Gamma(n+1) = N_n^2 \pi (2l^2)^{n+1} n! = 1, \tag{4.19}$$

$$N_n = \left(n!\pi(2l^2)^{n+1}\right)^{-1/2}. \tag{4.20}$$

Resulting in the general wave-function for the ground state

$$\psi_n(z, z^*) = \left(n!\pi(2l^2)^{n+1}\right)^{-1/2} z^n e^{\frac{-|z|^2}{4l^2}}. \tag{4.21}$$

Note first of all, that the probability density can be seen as a circumference that grows as n grows, this is also important because note that the average radius squared ($r^2 = x^2 + y^2 = zz^*$)can be written as

$$\overline{r^2} = N_n \pi \int e^{\frac{-t}{2l^2}} t^{n+1} dt = (n+1)2l^2 \tag{4.22}$$

Like has been said before the radius is bigger as n grows, and notice that n must be non negative. Otherwise the wave-function is not normalizable because of the singularity $z^{-|n|}$ at the origin. Also, only integer powers are allowed since a fractional power would lead to a multivalued wave-function.

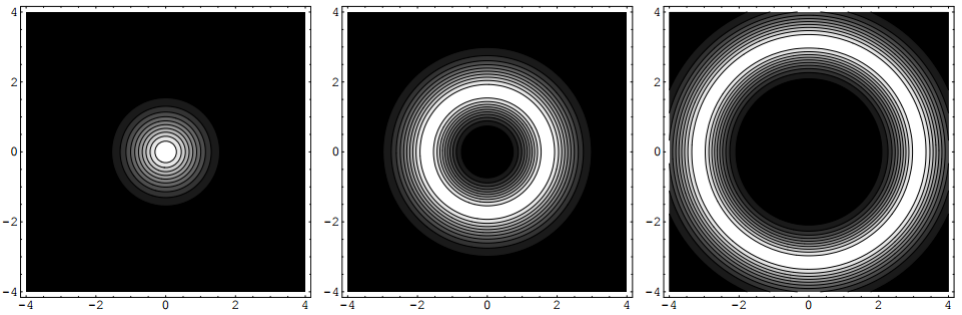


FIGURE 4.6: The ground state wave-function for the symmetric gauge with n=0,3,10 (Murayama, 2006)

One last thing to say about the choice of this gauge is that there exists degeneracy on the wave-function, but it will not be discussed here since the problem of finding the degree of degeneracy is rigorous. Murayama (2006) states that the level of degeneracy N is, $N = e\Phi/\hbar$, where Φ is the magnetic flux passing through the surface.

Translationally invariant gauge (Landau gauge): Using the choice of the landau gauge we have two options, it has to be said that the treatment for this one is the same as for the other, so this does not affect our solution. To start we will use the y translationally invariant gauge $\mathbf{A} = B(0, x)$ like has been done before it is useful to define the creation and annihilation operators

$$\hat{a} = \frac{l}{\sqrt{2}\hbar}\left(p_x + i\left(p_y - eBx\right)\right), \tag{4.23}$$

$$\hat{a} = -\frac{il}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \left(k_y - \frac{x}{l^2}\right)\right). \tag{4.24}$$

Similarly, the creation operator in the Landau gauge is

$$\hat{a}^\dagger = \frac{il}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \left(k_y - \frac{x}{l^2}\right)\right). \tag{4.25}$$

Repeating the process done before in the rotationally invariant gauge, and solving for $\hat{a}\,|0\rangle = 0$ in the $\langle x, y|$ basis we can get the ground state wave-function. This process will simplify in a very good way noticing that there is no y dependence as expected from the translational invariance, and hence we can take plane wave solutions. That is wave-functions with the form

$$\psi(x,y) = X(x)e^{ik_y y}. \tag{4.26}$$

Now as has been said solving $\hat{a}\,|0\rangle = 0$ with Equation 4.26

$$\langle x,y|\,\hat{a}\,|0\rangle = -\langle x,y|\,\frac{il}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \left(k_y - \frac{x}{l^2}\right)\right)|0\rangle = 0, \tag{4.27}$$

$$\left(\frac{\partial}{\partial x}X(x)e^{ik_y y} + \frac{1}{l^2}\left(k_y l^2 - x\right)X(x)e^{ik_y y}\right) = 0, \tag{4.28}$$

$$X(x) = Ae^{-\frac{1}{l^2}\left(x-k_y l^2\right)^2}. \tag{4.29}$$

Then

$$\psi_o(x,y) = Ae^{ik_y y - \frac{1}{l^2}\left(x-k_y l^2\right)^2}. \tag{4.30}$$

This is the ground state wave-function for the Landau gauge where A is a normalization constant to be found. The wave-functions are extended in y and localized in x. The localization remains unaffected under a gauge transformation. The wave function is a strip stretched along the y-axis, while is centered $x = k_y l^2$ as a Gaussian distribution. As in the former gauge the level of degeneracy N in this gauge is $N = e\Phi/\hbar$.

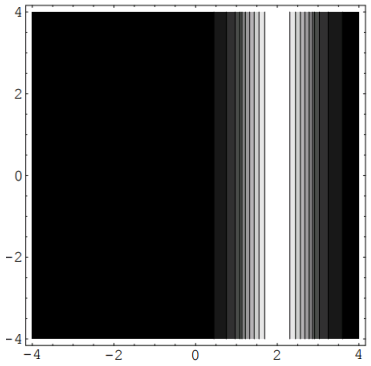


FIGURE 4.7: The ground state wave-function for the Landau gauge (Murayama, 2006)

4.1.4 The Quantization of the Hall Description

When solving this problem one sees that this system does not possesses an electric field, like the hall effect. In order to give the quantum treatment of the Hall effect one has to add the contribution of the electric field to the wavefunction. This can be accomplished in a very easy way. This problem will be solved on the landau gauge. For this particular case the use of creation an annihilation operators hinder the solution, since the electric field is non-zero. Therefore the easiest way to solve is to use the Schödinger equation. Thus the hamiltonian is the same as before but with the addition of the electric field contrbution, for simplicity suppose that there is electric field on the x-axis. Therefore

$$H = \frac{1}{2m_e}[p_x^2 + (p_y - eBx)^2] - eEx, \tag{4.31}$$

as was done before we might try a solution like Equation 4.26. The Schödinger equation then can be written as

$$\left(\frac{1}{2m_e}[p_x^2 + (\hbar k_y - eBx)^2] - eEx\right) X(x) = E_X X(x). \tag{4.32}$$

Here the energy is written E_X and for notation is different from the electric field. Making the substitution $D = -k_y l^2 + eEm_e l^4/\hbar^2$ in order to simplify the equation one gets

$$\left(\frac{1}{2m_e}p_x^2 + \frac{m_e\omega_c^2}{2}(x - D)\right) X(x) = \left(E_X + eED - \frac{m_e}{2}\left(\frac{E}{B}\right)^2\right) X(x), \tag{4.33}$$

by inspection, the left hand side of Equation 4.33 is the Schödinger equation for the harmonic oscillator centered on D, then the substitution made before can be interpreted as the center of the movement. The solution of this equation is given by any book on quantum mechanics. One knows that this solution is given in general by the Hermite polynomials (Auletta et al., 2009, p. 154), and it is

$$X(x) = \sqrt{\frac{1}{\pi^2 2^n n! l}} \exp\left\{-\frac{(x - D)^2}{2l^2}\right\} H_n\left(\frac{x - D}{l}\right). \tag{4.34}$$

So, the full solution to this problem is the combination of Equation 4.34 and Equation 4.26

$$\psi(x,y) = A\sqrt{\frac{1}{\pi^2 2^n n! l}}\exp\left\{ik_y y - \frac{(x-D)^2}{2l^2}\right\} H_n\left(\frac{x-D}{l}\right). \tag{4.35}$$

As can be seen from the right hand side of Equation 4.33 the energy values of this system are the same as in the harmonic potential case but with the addition of two terms. Therefore it can be written as

$$E = \hbar\omega_c\left(n+\frac{1}{2}\right) + eED - \frac{m_e}{2}\left(\frac{E}{B}\right)^2. \tag{4.36}$$

The last part of this description is to understand the hall conductivity under the aforementioned description. As it is well known the electric current density classically is $i = -n_e e v$. Here there wont be any speed on the x-axis since the mean value is zero. The y-axis speed is

$$\langle\psi|\, v_y\, |\psi\rangle = \langle\psi|\, \frac{1}{m_e}(p_y - eBx)\, |\psi\rangle\,, \tag{4.37}$$

After replacing as before

$$\langle\psi|\, v_y\, |\psi\rangle = \langle\psi|\, \frac{eB}{m_e}(D-x) - \frac{E}{B}\, |\psi\rangle = -\frac{E}{B}. \tag{4.38}$$

Hence taking in consideration the definition of electric current density, the fact that is a two dimensional system, and the definition of the conductivity tensor one arrives to

$$E_x\sigma_{xy} = -n_e e v_y. \tag{4.39}$$

Replacing the mean value of the speed in the y-axis one obtains

$$E_x\sigma_{xy} = -n_e e\frac{E_x}{B}. \tag{4.40}$$

The landau levels are infinitely degenerate if the area of the plane is infinite, but here we might restrict to an area of the larmor radius to solve this. So, the electronic density is just the number of electrons in the specific area of the plane $n_e = n/2\pi l^2$. Therefore one has

$$\sigma_{xy} = -\frac{ne}{2\pi l^2 B} = -\frac{ne}{\Phi_0}. \tag{4.41}$$

At this point is where the theory becames strange, since if we want to arrive to the cuantization of the hall effect one then has to use the definition of the quantum flux or flux unit which is given by $\Phi_0 = h/e$. The problem arises from the beginning of the description, since the hamiltonian given by Equation 4.31 represents a single particle hamiltonian, and from the experiments the electrons are interpreted as a gas. Therefore the description is only given for one electron. The other conceptual problem, is that here the number of electrons per landau levels will be n, that is the filling factor. Since there is only one electron the filling factor is then $\nu = n = 1$.

$$\sigma_{xy} = -n\frac{e^2}{h} = -\frac{e^2}{h}. \tag{4.42}$$

That is the quantization condition for the integer quantum hall effect but with a misconception in the quantization process, as mentioned before this is a approximation that possess an conceptual problem, the main solution to the integer quantum hall effect will not be treated here, since it presents other description that is not useful in the anyonic theory. But it description can be found in any textbook on the subject(Goerbig, 2009; Yoshioka, 2013).

4.2 The FQHE and the Anyonic Formalism

In the last section a simple and illuminating proof was given in order to find the quantization for the conductance (resistance). This proof possesses misconceptions in the explanation, since the description is made for only one particle even though the hall effect comprises N particles. The full understanding of the integer quantum hall effect is not discussed but it was found by Klitzing et al. (1980); Tsui et al. (1982). Here the main focus is the anyonic applications, and the only experimental application of the anyonic formalism is the fractional quantum hall effect. The understanding of this problem was something very difficult and proved to be challenging, since as told before the filling factor in this part is a fraction. R. Laughlin (1984); R. B. Laughlin (1983) proposed a form to describe the fractional quantum hall effect. This is named the variational approach. Namely, we assume an appropriate wave function $\Psi(\alpha)$ that contains a parameter α as a candidate for the ground state, and we calculate the expectation value of the Hamiltonian $\langle\Psi(\alpha)| H |\Psi(\alpha)\rangle / \langle\Psi(\alpha)|\Psi(\alpha)\rangle$. This value is always greater than or equal to the true ground-state energy. Therefore, by choosing the parameter α to minimize the value, we obtain an approximate ground state. A correct choice of the trial function is essential for this method to be successful. For the investigation of the fractional quantum hall effect, this method has proved to be quite powerful.

4.2.1 Laughlin's Trial Wavefunction

R. Laughlin proposed a wavefunction to understand the fractional quantum hall effect, by stating that first, the single-electron Hilbert space is restricted to the lowest Landau level, Second assume strong-magnetic-field. The single-electron wave function in the symmetric gauge is like Equation 4.17. In the restricted space, any wave function of N_e electrons can be expressed as a linear combination of Slater determinants composed of these single-electron wave functions. Therefore, it has the following form in general

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = f(z_1, \ldots . z_N) \exp\left\{ -\sum_i \frac{|z_i|^2}{4l^2} \right\}. \tag{4.43}$$

Here $f(z_1, \ldots . z_N)$ is again considered a polynomial in every variable z_i(there are N electrons). These restrictions should apply to all states, including the excited states, irrespective of the range of the interaction. In particular, an important piece of information, that the interaction between the electrons has a strong repulsive part at short distance, has not been taken into account. So let us utilize this information. The Coulomb interaction between electrons tends to separate electrons. It makes the wave function smaller when two electrons approach each other. Thus the wave function should be a function of the inter electron distances. Let us assume that the polynomial f is a function of the inter electron distances only. Then f can be written as

$$f(z_1, \ldots . z_N) = \prod_{i>j} g(z_i - z_j). \tag{4.44}$$

When this approximation is combined with the general restrictions, the functional form is fixed as $g(z) = N_q z^q$, where q must be an odd integer, and N_q is normalization constant. Thus we arrive at a trial function which contains only a single parameter q

$$\Psi_q(\mathbf{r}_1, \ldots, \mathbf{r}_N) = N_q \prod_{i>j} (z_i - z_j)^q \exp\left\{ -\sum_i \frac{|z_i|^2}{4l^2} \right\}. \tag{4.45}$$

Equation 4.45 is named the Laughlin's wavefunction, and it explains the collective states of electrons in the fractional quantum hall effect when $\nu = 1/q$, specifically when $q = 3$ the results are very approximate to the experimental results.

4.2.2 Quasi-particles and Quasi-holes

There is an interesting connection of the Laughlin trial wave function with the two dimensional gas of classical particles. Many of the phenomenological consequences of the trial function are in fact obtained from this plasma analogy. In particular, the quantum probability density $|\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)|^2$ can be interpreted as the probability distribution function of a classical statistical mechanics problem. If we take

$$|\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)|^2 = e^{-\Omega}, \tag{4.46}$$

we obtain by Equation 4.45 that the parameter on the exponential is

$$\Omega = -2q \sum_{i<j} ln|z_i - z_j| + \frac{1}{2l^2} \sum_k |z_k|^2. \tag{4.47}$$

This is the potential energy of a one component plasma (Khare, 2005, p. 267). Now it is well known that a two-dimensional one-component plasma of particles with charge q' moving in a neutralizing background of density ρ' is characterized by a potential energy given by

$$\Omega = -q'^2 \sum_{i<j} ln|z_i - z_j| + \frac{\pi \rho' q'^2}{2} \sum_k |z_k|^2. \tag{4.48}$$

By direct comparison between Equation 4.47 and Equation 4.48 one can see that

$$\rho' = \frac{1}{2\pi q l^2}. \tag{4.49}$$

One knows that the state is a liquid because the temperature of the equivalent plasma is sufficient to melt it. The potential energy Ω is minimized (i.e. $|\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)|^2$ is maximized) by the uniform distribution of charged particles and is electrically neutral everywhere when the average density of electrons n_e exactly equals ρ' i.e, when the filling fraction $\nu = 1/q$. These arguments provide good justification for identifying the Laughlin wave function as the ground state of the fractional quantum Hall state at filling fraction $\nu = 1/q$. These arguments suggest that the Laughlin wave function represents an incompressible quantum liquid and that incompressibility is, in a sense, the essence of the fractional quantum Hall effect. When n_e deviates from ρ' by a tiny amount, that is one adds a single particle to the ensemble, any deviation from ρ' lead to the creation of localized quasi-particle or quasi-hole excitations. Halperin (1983) pointed out

that if, in Laughlin's state, we fix the positions of all electronsexcept one (z_1) and move that electron around a closed loop, avoiding other electrons in the system, the phase of the wave function changes by $\Delta\phi = 2\pi q\Phi/h$. With this in mind Halperin pointed out that Laughlin's wave function has precisely q vortices at each electron position and no other wasted vortices elsewhere in the sample. This quasi-particles are then treated as anyons. The quasi-hole and the quasi-particle excitations at filling $\nu = 1/q$ are approximately described by the wave functions

$$\Psi_q^{+z_\alpha}(\mathbf{r}_1,\ldots,\mathbf{r}_N) = N_+ \prod_i (z_i - z_\alpha) \prod_{j<k} (z_j - z_k)^q \exp\left\{-\sum_k \frac{|z_k|^2}{4l^2}\right\}, \tag{4.50}$$

$$\Psi_q^{-z_\alpha}(\mathbf{r}_1,\ldots,\mathbf{r}_N) = N_- \prod_i \left(\frac{\partial}{\partial z_i} - \frac{\overline{z}_\alpha}{4l^2}\right) \prod_{j<k} (z_j - z_k)^q \exp\left\{-\sum_k \frac{|z_k|^2}{4l^2}\right\}. \tag{4.51}$$

The $N_\pm$ is a normalization constant, and z_α is interpreted as the center of the excitation. Now we need to compute the charge and the statistics that this quasi-particles(quasi-holes) have.

4.2.3 Charge of quasi-holes

The calculation of the charge and the statistics is very important since it will give the basis to understand the application of anyonic statistics in the fractional quantum hall effect. In order to find the charge of an excitation one have to use the concept of berry phase. According to D. Arovas, Schrieffer, and Wilczek (1984). To determine the quasi-particle charge e^*, we calculate the change of phase γ of $\Psi_q^{+z_\alpha}$ as z_α adiabatically moves around a circle of radius R enclosing flux Φ. The adiabatic theorem states that if we slowly vary the parameters in a Hamiltonian, then a state occupying an isolated energy level remains fully in that level and there are no quantum jumps. Berry made a profound contribution concerning the change in the phase of the wave function if the Hamiltonian, after a sequence of adiabatic changes, returns to its starting value (Berry, 1984). This phase is given by

$$\gamma = -i \int_{t_0}^{t_1} dt \, \langle \Psi(t)| \frac{\mathrm{d}}{\mathrm{d}t} |\Psi(t)\rangle \,. \tag{4.52}$$

For simplicity let us compute the berry phase for a quasi-hole[1]. The computing is made under the assumption that the quasi-hole is slowly transported around a closed loop Γ so that z_α

[1] Since is the same as for the quasi-particle, but differing by a sign

becomes a time-dependent parameter and the adiabatic approximation can be used. Since the entire time dependence of the problem comes solely from z_α. Therefore we have for a quasi-hole

$$\gamma = -i \int_{t_0}^{t_1} dt \left\langle \Psi_q^{+z_\alpha}(t) \right| \sum_i \frac{\mathrm{d}}{\mathrm{d}t}(ln|z_i - z_\alpha|) \left| \Psi_q^{+z_\alpha}(t) \right\rangle . \tag{4.53}$$

Here the integral over t is from initial time t_0 when the quasi-hole begins its tour, to the final time t_1 when it returns to its original position after traversing a non-trivial loop Γ in the anti-clockwise direction. This expression for γ can be written in a more transparent way by noting that the electron density ρ in the quasi-hole state is $\rho^{+z_\alpha}(z) = \left\langle \Psi_q^{+z_\alpha} \right| \sum_i \delta(z - z_i) \left| \Psi_q^{+z_\alpha} \right\rangle$ (D. Arovas et al., 1984). Thus

$$\gamma = -i \int_{t_0}^{t_1} dt \int d^2z \left(\frac{\mathrm{d}}{\mathrm{d}t}(ln|z_i - z_\alpha|) \right) \rho^{+z_\alpha}(z), \tag{4.54}$$

$$\gamma = -i \int d^2z \int_{t_0}^{t_1} dt \left(\frac{dz_\alpha(t)}{dt} \right) \frac{1}{z_\alpha(t) - z} \rho^{+z_\alpha}(z), \tag{4.55}$$

$$\gamma = -i \int d^2z \oint_\Gamma dz_\alpha \frac{1}{z_\alpha - z} \rho^{+z_\alpha}(z), \tag{4.56}$$

$$\gamma = 2\pi \int d^2z \rho^{+z_\alpha}(z) = 2\pi N_\Gamma. \tag{4.57}$$

Here N_Γ represents the number of electrons that are inside the region of the loop. Now by direct comparison with the phase acquired by a particle with charge e^* under the motion around a solenoid (Equation 3.2) one obtains that

$$2\pi N_\Gamma = \frac{e^* \Phi_\Gamma}{\hbar}. \tag{4.58}$$

Thus

$$e^* = \frac{h N_\Gamma}{\Phi_\Gamma}. \tag{4.59}$$

In order to obtain the desired result one can use the precise definition of the filling factor(Chakraborty & Pietiläinen, 2013, p. 6). That is $\nu = N\Phi_0/\Phi$, then for our specific case one has

$$\frac{1}{q} = \frac{N_\Gamma \Phi_0}{\Phi_\Gamma}. \tag{4.60}$$

Therefore the charge of the quasi-hole is

$$e^* = \frac{e}{q} = \nu\, e. \tag{4.61}$$

For obvious reasons the charge of the quasi-particle is the same Equation 4.61 but with the opposite sign. For the moment we can see that the charge of this quasi-particles(quasi-holes) is less than the charge of the electron, but that is not an insight that they are anyons. In fact if we want to say that first we have to find the statistics that two quasi-particles(quasi-holes) obeys under a symmetry exchange.

4.2.4 statistics of quasi-holes

The main description of anyons deals with particles in low dimensional systems, but not only that what characterizes anyons, is the fact that under a symmetry exchange of two anyons, the wavefunction changes by a phase which was already found. Now in order to understand the main application of anyons in the description of the fractional quantum hall effect one can allow for an exchange of two quasi-particles(quasi-holes). Again for simplicity the case of quasi-holes will be treated. Let us then consider a state characterized by the presence of two quasi-hole excitations, one located in the point z_α and the other located in the point z_β. If we neglect any interaction between the two quasi-holes, the wavefunction turns out to be

$$\Psi_q^{+z_\alpha,+z_\beta}(\mathbf{r}_1,\ldots,\mathbf{r}_N) = N_{\alpha\beta}\prod_i (z_i - z_\alpha)(z_i - z_\beta)\prod_{j<k}(z_j - z_k)^q \exp\left\{-\sum_k \frac{|z_k|^2}{4l^2}\right\}. \tag{4.62}$$

Now let us assume that the quasi-hole in z_α is adiabatically moved on a full loop Γ, while the quasi-hole in z_β is kept fixed. During this process the wavefunction in Equation 4.62 acquires a Berry phase γ given again by Equation 4.52 that can be calculated as before

$$\gamma = -i \int_{t_0}^{t_1} dt \left\langle \Psi_q^{+z_\alpha,+z_\beta}(t) \right| \sum_i \frac{\mathrm{d}}{\mathrm{d}t}(ln|z_i - z_\alpha|) \frac{\mathrm{d}}{\mathrm{d}t}(ln|z_i - z_\beta|) \left| \Psi_q^{+z_\alpha,+z_\beta}(t) \right\rangle . \tag{4.63}$$

In taking the electronic density for the two quasi-hole state as $\rho^{+z_\alpha,+z_\beta}(z) = \left\langle \Psi_q^{+z_\alpha,+z_\beta} \right| \sum_i \delta(z - z_i) \left| \Psi_q^{+z_\alpha,+z_\beta} \right\rangle$ and skipping the intermediate steps which are exactly the same, we have the overall result for the berry phase

$$\gamma = 2\pi \int d^2z \, \rho^{+z_\alpha,+z_\beta}(z). \tag{4.64}$$

Clearly, if in its motion z_α does not encircle z_β, then the integral has the same value as in Equation 4.57 so that in that case the two quasi-holes are not exchanged, then we have that the loop Γ is homotopically equivalent to any loop defined on the configuration space[2]. On the other hand, if the loop Γ traversed by z_α contains z_β, then is homotopically non-trivial (the space is multiply-connected), then we will get a different value for the aforementioned integral in Equation 4.64. That value is (Khare, 2005; Lerda, 2008)

$$\gamma = 2\pi \int d^2z \, \rho^{+z_\alpha,+z_\beta}(z) = 2\pi \left(N_\Gamma - \frac{1}{q} \right). \tag{4.65}$$

If a quasi-hole is inside the loop Γ, then the number of electrons have to be diminished by the fraction $1/q$ that is needed to build the quasi-hole. Therefore we can conclude that when a quasi-hole at z_α encircles another quasi-hole at z_β, the wavefunction acquires an extra phase given by $\exp\{-i\Delta\gamma\} = \exp\{i2\pi/q\}$. As one quasi-hole follows a path around the other quasi-hole this can be thought as a double exchange of particles. Also as the space is multiply connected the phase can be equated to that in Equation 3.22 but with one difference and that is, that the azimuth angle used there was for a single exchange, here as there are two exchanges the angle goes from 0 to 2π. Therefore

$$\exp\left\{ i\Upsilon \int_0^{2\pi} d\varphi \right\} = \exp\{i2\pi/q\} \tag{4.66}$$

Which gives us

[2]Imagine that one particle moves on a circle far from other particle, then any loop which does not encircle the second particle can be continuously deformed into a point, therefore the loop is homotopically equivalent to a point

$$\Upsilon = \frac{1}{q} = \nu. \tag{4.67}$$

We can conclude that the statistics obeyed by quasi-holes(quasi-particles) is the anyonic statistics, therefore the quasi-holes(quasi-particles) are anyons with fractional charge. As the fractional hall effect for filling factors of the form in Equation 4.67 is experimentally well studied, the theory of anyons then can be given a truth value as physically one can measure the statistical factor associated with anyons, by measuring the filling factor of a determined fractional quantum hall effect.

4.3 FQHE Remarks

As was proved in the entire chapter, the fractional quantum hall effect proved to be a difficult phenomena to understand, at that time the full understanding was still a complete mystery since the solution proposed by Laughlin is not the exact solution. Up to now the full solution is already known and the theory of Laughllin has already been validated(F. Haldane & Rezayi, 1985; Levesque, Weis, & MacDonald, 1984; Morf & Halperin, 1986, 1987; Yoshioka, 1984; Yoshioka, Halperin, & Lee, 1983; Yoshioka & Lee, 1983). Then the understanding of the fractional quantum hall effect in terms of the anyonic formalism is also validated. The complete understanding of the fractional quantum hall effect for filling factors of the form $\nu = \frac{p}{q}$ has also been understood by the proposal of the hierarchy model(F. D. M. Haldane, 1983; Halperin, 1983, 1984). Also this model retrieves the anyonic phase so it study will not be developed here, therefore we have proved that the anyonic statistics is not only a mathematical tool, but a theoretical way to understand a physical model.

Chapter 5

Conclusion

In this final part of this work the full list of properties that anyons possesses will be shown. Since the study up to now is entirelly devoted to the development of the full theory of anyons the physical properties were not showed. Also the main applications of the anyonic statistics will be told since this theory has also been proposed to understand new types of phenomena that arise in the actuality. And to end the final remarks will be told.

5.1 Properties of anyons

By reviewing the study up to now we may want to state the main properties of anyons, some properties has been showed while others had been postponed for this discussion.

1. Anyons are necessarily two dimensional particles, that possesses an associated flux, but not only this anyons are quasi-particles(quasi-holes) and therefore its physical existence is limited to two dimensional gases.

2. Anyons arise in the context of multiply-connected spaces, thus the topology of a two dimensional system is related to the physical existence of an anyon.

3. Anyons must necessarily violate the discrete symmetries of parity $\mathcal{P}$ and time reversal invariance $\mathcal{T}$, this is clearly seen as since in the braid group one can difference between a clockwise winding and a counter-clockwise winding, and the fact that braid diagrams are parametrized in time.

4. Anyons are sort of in between the bosons and the fermions i.e. the repulsion in states that fermions "fell" increases as Υ gets closer to 1, also the repulsion disappears when Υ gets closer to 0. This can be seen better in Equation 3.23

5. Anyons posses as a consecuence of a non-trivial phase a very different spin, since $so(2)$ is abelian, and the braid group is also abelian, thus anyons definitely are new type of particles.

6. Anyons necessarily must exchange their positions according to the adiabatic theorem since they obey the berry's phase, thus any exchange interaction is entirely adiabatic.

7. Anyons relate entirely with the Aharonov-Bohm effect in the phase acquired, this since the configuration space for a charged particle in the aforementioned effect is multiply-connected.

8. Anyons must necessarily be identical particles, since the configuration space leads to the one dimensional representations of the braid group.

9. The anyonic gauge is indeed some sort of symmetric gauge but with one difference in the statistical parameter.

10. The complete study of N anyons is an impossible task, furthermore even though we can develop the tools to describe entirely this system the main application of it seem unpractical.

Anyons propose an interesting way to study the flatland that is low dimensional systems, and this study of course is not completed, recent discoveries have shown that the order of the representation in the braid group matters, and then the representation lead to non-abelian anyons. In recent years, the anyonic statistics has been proposed as a solution to a variety of problems in physics. As has been told anyons are a proposal to high temperature superconductors (Chen et al., 1989; Wilczek, 1990), pitifully although the mathematics has been studied no clear solution has been given up to now that can be experimentally verified. Also anyons are proposed as a new solution to quantum computing (Averin & Goldman, 2001; Comtet et al., 1989; Freedman et al., 2003; Kitaev, 2003; Mochon, 2003; Williams, 1999). As quantum computing itself is in its infancy, anyonic models of quantum computers are still a theoretical development.

Appendix A

The Braid Group

As can be seen in describing an anyonic system one has to deal with the representation of the Braid group of order 2, so a basic treatment of braid groups will be given here with the aim of support the mathematical description given on chapter 2. This introduction will follow an entirely non-rigurous approach to the subject, and it is based on Artin (1947a, 1947b); Gassner (1961); Kassel and Turaev (2008); Lin (2004).

Definition A.1 (**Artin Braid Group**). *The Artin Braid Group B_N is the group generated by $N-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{N-1}$, and the "Braid relations"*

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all $i, j = 1, 2, \ldots, N-1$ with $|i-j| \geq 2$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for all $i, j = 1, 2, \ldots, N-2$.

By definition, $B_1 = \{1\}$ is a trivial group. The group B_2 is generated by a single generator σ_1 and an empty set of relations. This is an infinite cyclic group. The groups B_N with $N \geq 3$ are nonabelian.

Lemma A.2. *if $s_1, s_1, \ldots, s_{N-1}$ are elements of a group G satisfying the Braid relations, then there is a unique homomorphism $\chi : B_N \to G$ such that $s_i = \chi(\sigma_i)$ for all $i = 1, 2, \ldots, N-1$.*

To understand better the braid group it is necessary to give a geometrical interpretation of the braids, so it will be usefull to make the next definition in order to link the geometry of the braids and the group form of it

Definition A.3 (**Geometrical Braid**). *A geometrical Braid on $N \geq 1$ strings is a set $b \subset \mathbb{R}^2 \times I$ formed by N disjoint topological intervals called the strings of b such that the projection $\mathbb{R}^2 \times I \to I$ maps each string homeomorphically onto I and*

$$b \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0),(2,0,0),\ldots,(N,0,0)\}$$

$$b \cap (\mathbb{R}^2 \times \{1\}) = \{(1,0,1),(2,0,1),\ldots,(N,0,1)\}$$

It is clear that every string of b meets each plane $\mathbb{R}^2 \times \{t\}$ with $t \in I$ in exactly one point and connects a point $(i,0,0)$ to a point $(s(i),0,1)$, where $i, s(i) \in 1,2,\ldots,N$. The sequence $(s(1),s(2),...,s(N))$ is a permutation of the set $\{1,2,...,N\}$ called the underlying permutation of b. To see better this imagine a plane with x,y coordinates in $\mathbb{R}^2$ the braid $(1,2,3,4)$ is permuted on $(1,3,2,4)$, on a figure this can be seen as

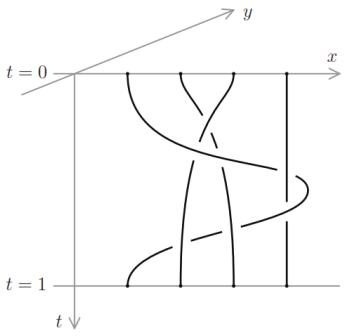


FIGURE A.1: Permutation of the braid $(1,2,3,4)$ on $(1,3,2,4)$ (Kassel & Turaev, 2008, P. 5)

To specify a geometric braid, one can draw its projection to $\mathbb{R} \times \{0\} \times I$ along the second coordinate and indicate which string goes "under" the other one at each crossing point, these considerations lead to the notion of braid diagram.

Definition A.4 (**Braid Diagram**). *A braid diagram on N strands is a set $D \subset \mathbb{R} \times I$ split as a union of N topological intervals called the strands of D such that the following three conditions are met:*

- *The projection $\mathbb{R} \times I \to I$ maps each strand homeomorphically onto I*

- *Every point of* $\{1, 2, ..., N\} \times \{0, 1\}$ *is the endpoint of a unique strand.*
- *Every point of* $\mathbb{R} \times I$ *belongs to at most two strands. At each intersection point of two strands, these strands meet transversely, and one of them is distinguished and said to be* ***undergoing****, the other strand being* ***overgoing****.*

This transforms D into a geometric braid on N strings. Its isotopy class is a well-defined braid presented by D. This braid is denoted by $\beta(D)$. For instance, the braid diagram in Figure A.1 presents the braid drawn in Figure A.2.

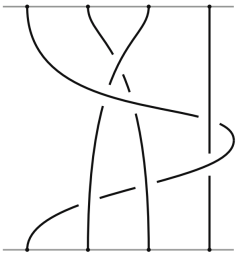


FIGURE A.2: Braid Diagram representing the braid permutation $(1, 2, 3, 4)$ on $(1, 3, 2, 4)$ (Kassel & Turaev, 2008, P. 9)

Denote by $\mathcal{B}_N$ the set of braids on N strings with multiplication indicated by the twisting of a string over another. The next lemma implies that $\mathcal{B}_N$ is a group, and as a group it preserves the braid relations.

Lemma A.5. *Each* $\beta \in \mathcal{B}_N$ *has a two-sided inverse* $\beta^{-1} \in \mathcal{B}_N$

Lemma A.6. *The elements* $\sigma_1^+, \ldots, \sigma_{N-1}^+ \in \mathcal{B}_N$ *are the generators in* $\mathcal{B}_N$*, they preserve the braid relations and represent the crossing of strings counter-clockwise in the braid diagram*

$$\sigma_i^+ \sigma_j^+ = \sigma_j^+ \sigma_i^+$$

for all $i, j = 1, 2, \ldots, N-1$ *with* $|i - j| \geq 2$ *and*

$$\sigma_i^+ \sigma_{i+1}^+ \sigma_i^+ = \sigma_{i+1}^+ \sigma_i^+ \sigma_{i+1}^+$$

for all $i, j = 1, 2, \ldots, N-2$.

Lemma A.7. *Any abelian homomorphism* χ_θ *of* $\mathcal{B}_N$ *is cyclic and satisfies* $\chi_\theta(\sigma_1^+) = \chi_\theta(\sigma_2^+) = \cdots = \chi_\theta(\sigma_{N-1}^+)$

to end the discussion of the braid group, we shall understand that for this project it will be used the braid diagram, so for the sake of simplicity $\sigma_i^+ = \sigma_i$, the inverse in this group represent a clockwise winding on the braid diagram and it will be written $\sigma_i^- = \sigma_i^{-1}$,

References

Abbott, E. A. (2006). *Flatland: A romance of many dimensions*. OUP Oxford.

Aharonov, Y., & Bohm, D. (1959). Significance of electromagnetic potentials in the quantum theory. *Physical Review*, *115*(3), 485.

Aharonov, Y., & Bohm, D. (1961). Further considerations on electromagnetic potentials in the quantum theory. *Physical Review*, *123*(4), 1511.

Arfken, G. B., Weber, H. J., & Harris, F. E. (2011). *Mathematical methods for physicists: a comprehensive guide*. Academic press.

Arovas, D., Schrieffer, J. R., & Wilczek, F. (1984). Fractional statistics and the quantum hall effect. *Physical review letters*, *53*(7), 722.

Arovas, D. P., Schrieffer, R., Wilczek, F., & Zee, A. (1985). Statistical mechanics of anyons. *Nuclear Physics B*, *251*, 117–126.

Artin, E. (1947a). Braids and permutations. *Annals of Mathematics*, 643–649.

Artin, E. (1947b). Theory of braids. *Annals of Mathematics*, 101–126.

Auletta, G., Fortunato, M., & Parisi, G. (2009). *Quantum mechanics*. Cambridge University Press.

Averin, D., & Goldman, V. (2001). Quantum computation with quasiparticles of the fractional quantum hall effect. *Solid state communications*, *121*(1), 25–28.

Batelaan, H., & Tonomura, A. (2009). The aharonov-bohm effects: Variations on a subtle theme. *Physics Today*, *62*(9), 32.

Berry, M. V. (1984). Quantal phase factors accompanying adiabatic changes. In *Proceedings of the royal society of london a: Mathematical, physical and engineering sciences* (Vol. 392, pp. 45–57).

Bhaduri, R., Bhalerao, R., Khare, A., Law, J., & Murthy, M. (1991). Semiclassical two-and three-anyon partition functions. *Physical review letters*, *66*(5), 523.

Bose, S. N. (1924). Plancks gesetz und lichtquantenhypothese. *Z. phys*, *26*(3), 178.

Chakraborty, T., & Pietiläinen, P. (2013). *The quantum hall effects: integral and fractional* (Vol. 85). Springer Science & Business Media.

Chen, Y.-H., Wilczek, F., Witten, E., & Halperin, B. I. (1989). On anyon superconductivity. *International Journal of Modern Physics B*, *3*(07), 1001–1067.

Comtet, A., Georgelin, Y., & Ouvry, S. (1989). Statistical aspects of the anyon model. *Journal of Physics A: Mathematical and General*, *22*(18), 3917.

Comtet, A., Jolicoeur, T., Ouvry, S., & David, F. (1999). Topological aspects of low dimensional systems. In *Topological aspects of low dimensional systems* (Vol. 1).

Dirac, P. A. M. (1926). On the theory of quantum mechanics. In *Proceedings of the royal society of london a: Mathematical, physical and engineering sciences* (Vol. 112, pp. 661–677).

Dirac, P. A. M. (1981). *The principles of quantum mechanics* (No. 27). Oxford university press.

Dowker, J. S. (1985). Remarks on non-standard statistics. *Journal of Physics A: Mathematical and General*, *18*(18), 3521.

Einstein, A. (1924). *Quantentheorie des einatomigen idealen gases.* Preussische Akademie der Wissenschaften.

Fadell, E., & Neuwirth, L. (1962). Configuration spaces. *Mathematica Scandinavica*, *10*, 111–118.

Fadell, E., Van Buskirk, J., et al. (1962). The braid groups of e^2 and s^2. *Duke Mathematical Journal*, *29*(2), 243–257.

Fasano, A., & Marmi, S. (2006). *Analytical mechanics: an introduction.* Oxford University Press.

Fermi, E. (1926). Sulla quantizzazione del gas perfetto monoatomico. *Rendiconti Lincei*, *3*, 145–149.

Feynman, R. P., Hibbs, A. R., & Styer, D. (2010). *Quantum mechanics and path integrals.* Dover Publications.

Fierz, M. (1939). Über die relativistische theorie kräftefreier teilchen mit beliebigem spin. *Helv. Phys. Acta*, *12*(3).

for91days.com. (2016, October). *Shibuya crossing tokyo 03.* Online. Retrieved from http://for91days.com/photos/Tokyo/Shibuya%20Crossing%20Tokyo/Shibuya%20Crossing%20Tokyo%20%2003%2020140504%20www.for91days.com%20DSC00492.JPG

Fox, R., & Neuwirth, L. (1962). The braid groups. *Mathematica Scandinavica*, *10*, 119–126.

Freedman, M., Kitaev, A., Larsen, M., & Wang, Z. (2003). Topological quantum computation. *Bulletin of the American Mathematical Society*, *40*(1), 31–38.

Gallier, J. (2011). *Geometric methods and applications: for computer science and engineering* (Vol. 38). Springer Science & Business Media.

Gassner, B. J. (1961). On braid groups. In *Abhandlungen aus dem mathematischen seminar*

der universität hamburg (Vol. 25, pp. 10–22).

Gibbs, J. W. (1878). On the equilibrium of heterogeneous substances. *American Journal of Science*(96), 441–458.

Goerbig, M. O. (2009). Quantum hall effects. *arXiv preprint arXiv:0909.1998*.

Haldane, F., & Rezayi, E. (1985). Finite-size studies of the incompressible state of the fractionally quantized hall effect and its excitations. *Physical review letters*, *54*(3), 237.

Haldane, F. D. M. (1983). Fractional quantization of the hall effect: a hierarchy of incompressible quantum fluid states. *Physical Review Letters*, *51*(7), 605.

Halperin, B. I. (1983). Theory of the quantized hall conductance. *Helvetica Physica Acta*, *56*(1-3), 75–102.

Halperin, B. I. (1984). Statistics of quasiparticles and the hierarchy of fractional quantized hall states. *Physical Review Letters*, *52*(18), 1583.

Heisenberg, W. (1985). Mehrkörperproblem und resonanz in der quantenmechanik. In *Original scientific papers wissenschaftliche originalarbeiten* (pp. 456–471). Springer.

Johnson, M., & Canright, G. (1990). Anyons in a magnetic field. *Physical Review B*, *41*(10), 6870.

js.postbit.com. (2016, October). *javascript canvas particles waves demo 1024.* Online. Retrieved from http://js.postbit.com/upload/63/post-20120801/javascript-canvas-particles-waves-demo-1024-postbit-36457.jpg

Juthe, A. (2005). Argument by analogy. *Argumentation*, *19*(1), 1–27.

Karlhede, A., & Westerberg, E. (1992). Anyons in a magnetic field. *International Journal of Modern Physics B*, *6*(10), 1595–1621.

Kassel, C., & Turaev, V. (2008). *Braid groups* (No. 247). Springer Science & Business Media.

Khare, A. (2005). *Fractional statistics and quantum theory* (2nd ed.). World Scientific.

Kitaev, A. Y. (2003). Fault-tolerant quantum computation by anyons. *Annals of Physics*, *303*(1), 2–30.

Klitzing, K. v., Dorda, G., & Pepper, M. (1980). New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance. *Physical Review Letters*, *45*(6), 494.

Laidlaw, M. G., & DeWitt, C. M. (1971). Feynman functional integrals for systems of indistinguishable particles. *Physical Review D*, *3*(6), 1375.

Laughlin, R. (1984). Primitive and composite ground states in the fractional quantum hall effect. *Surface Science*, *142*(1-3), 163–172.

Laughlin, R. B. (1983). Anomalous quantum hall effect: an incompressible quantum fluid with fractionally charged excitations. *Physical Review Letters*, *50*(18), 1395.

Laughlin, R. B. (1988). Superconducting ground state of noninteracting particles obeying

fractional statistics. *Physical review letters*, *60*(25), 2677.

Le Bellac, M. (2011). *Quantum physics*. Cambridge University Press.

Leinaas, J. M., & Myrheim, J. (1977). On the theory of identical particles. *Il Nuovo Cimento B (1971-1996)*, *37*(1), 1–23.

Lerda, A. (2008). *Anyons: Quantum mechanics of particles with fractional statistics* (No. 14). Springer Science & Business Media.

Levesque, D., Weis, J., & MacDonald, A. (1984). Crystallization of the incompressible quantum-fluid state of a two-dimensional electron gas in a strong magnetic field. *Physical Review B*, *30*(2), 1056.

Lin, V. (2004). Braids and permutations. *arXiv preprint math/0404528*.

Mochon, C. (2003). Anyons from nonsolvable finite groups are sufficient for universal quantum computation. *Physical Review A*, *67*(2), 022315.

Morf, R., & Halperin, B. (1986). Monte carlo evaluation of trial wave functions for the fractional quantized hall effect: disk geometry. *Physical Review B*, *33*(4), 2221.

Morf, R., & Halperin, B. (1987). Monte carlo evaluation of trial wavefunctions for the fractional quantized hall effect: Spherical geometry. *Zeitschrift für Physik B Condensed Matter*, *68*(2-3), 391–406.

Munkres, J. (2000). *Topology* (2nd ed.). Prentice Hall, Inc.

Murayama, H. (2006, February). *Landau levels*. Online. Retrieved from http://hitoshi.berkeley.edu/221A-F04/Landau.pdf

Neumann, J. V. (1931). Die eindeutigkeit der schrödingerschen operatoren. *Mathematische Annalen*, *104*(1), 570–578.

Neumann, J. V. (1932). Uber einen satz von herrn m.h. stone. *Annals of Mathematics*, 567–573.

Neumann, J. V. (1955). *Mathematical foundations of quantum mechanics* (No. 2). Princeton university press.

Patterson, J. D., & Bailey, B. C. (2007). *Solid-state physics: introduction to the theory*. Springer Science & Business Media.

Pauli, W. (1940). The connection between spin and statistics. *Physical Review*, *58*(8), 716.

Reyes, A. F. (2006). *On the geometry of the spin-statistics connection in quantum mechanics* (Unpublished doctoral dissertation). Johannes Gutenberg-Universitat Mainz.

Salinas, S. (2013). *Introduction to statistical physics*. Springer Science & Business Media.

Scheck, F. (2007). *Quantum physics*. Springer.

Schulman, L. (1968). A path integral for spin. *Physical Review*, *176*(5), 1558.

Stone, M. H. (1930). Linear transformations in hilbert space iii. operational methods and group theory. *Proceedings of the National Academy of Sciences*, *16*(2), 172–175.

Stone, M. H. (1932). On one-parameter unitary groups in hilbert space. *Annals of Mathematics*, 643–648.

Tsui, D. C., Stormer, H. L., & Gossard, A. C. (1982). Two-dimensional magnetotransport in the extreme quantum limit. *Physical Review Letters*, *48*(22), 1559.

Tung, W.-K. (1985). *Group theory in physics*. World Scientific.

Verçin, A. (1991). Two anyons in a static, uniform magnetic field. exact solution. *Physics Letters B*, *260*(1-2), 120–124.

Von Klitzing, K. (1986). The quantized hall effect. *Reviews of Modern Physics*, *58*(3), 519.

Wikipedia. (2016, August). *Hall effect.* online. Retrieved from https://upload.wikimedia.org/wikipedia/en/1/19/Hall_Effect_Measurement_Setup_for_Electrons.png

Wilczek, F. (1982a). Magnetic flux, angular momentum, and statistics. *Physical Review Letters*, *48*(17), 1144.

Wilczek, F. (1982b). Quantum mechanics of fractional-spin particles. *Physical Review Letters*, *49*(14), 957.

Wilczek, F. (1990). *Fractional statistics and anyon superconductivity* (Vol. 5). World Scientific.

Wilczek, F., & Shapere, A. (1989). *Geometric phases in physics* (Vol. 5). World Scientific.

Williams, C. P. (1999). *Quantum computing and quantum communications: First nasa international conference, qcqc'98, palm springs, california, usa, february 17-20, 1998, selected papers* (Vol. 1509). Springer Science & Business Media.

Wu, Y.-S. (1984a). General theory for quantum statistics in two dimensions. *Physical Review Letters*, *52*(24), 2103.

Wu, Y.-S. (1984b). Multiparticle quantum mechanics obeying fractional statistics. *Physical Review Letters*, *53*(2), 111.

Yoshioka, D. (1984). Ground state of the two-dimensional charged particles in a strong magnetic field and the fractional quantum hall effect. *Physical Review B*, *29*(12), 6833.

Yoshioka, D. (2013). *The quantum hall effect* (Vol. 133). Springer Science & Business Media.

Yoshioka, D., Halperin, B., & Lee, P. (1983). Ground state of two-dimensional electrons in strong magnetic fields and 1 3 quantized hall effect. *Physical review letters*, *50*(16), 1219.

Yoshioka, D., & Lee, P. (1983). Ground-state energy of a two-dimensional charge-density-wave state in a strong magnetic field. *Physical Review B*, *27*(8), 4986.